A Note on Core Logic

Peter Smith

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I'll take it, for the purposes of this Note, that the shape of Neil Tennant's preferred Intuitionistic Relevant Logic (**IR**) – latterly rebranded as 'Core Logic' – is familiar at least in broad outline (for some headlines, see his 'Relevance in Reasoning' in Shapiro's *Oxford Handbook of Philosophy of Mathematics and Logic*). I'm concerned here with three arguments offered in a forthcoming paper by Joseph Vidal-Rosset, 'Why Intuitionistic Relevant Logic Cannot Be a Core Logic'. I'll argue these supposed objections are either based on misunderstandings or are little more than incredulous stares.

1 The meaning of '$\vee$'

Tennant's preferred $\vee$E rule can be schematically put like this:

$$\frac{A \vee B \qquad \begin{matrix}[A] \\ \vdots \\ \bot/C\end{matrix} \qquad \begin{matrix}[B] \\ \vdots \\ \bot/C\end{matrix}}{\bot/C}$$

where if both the subproofs end in $\bot$ so does the whole proof, but if at least one subproof ends in C, then the whole proof ends in C. Tennant claims, in a passage quoted by Vidal-Rosset, that this liberalization of proof by cases compared with a more standard version is entirely natural, given how we reason informally:

> Suppose one is told that $A \vee B$ holds, along with certain other assumptions X, and one is required to prove that C follows from the combined assumptions $X, A \vee B$. If one assumes A and discovers that it is inconsistent with X, one simply stops one's investigation of that case, and turns to the case B. If C follows in the latter case, one concludes C as required. One does not go back to the conclusion of absurdity in the first case, and artificially dress it up with an application of the absurdity rule so as to make it also "yield" the conclusion C.

But Vidal-Rosset objects that Tennant's $\vee$E rule "changes the established meaning of $\vee$ when $\vee$ is on the left of the turnstile." Why so? He writes:

> $A \vee B \vdash C$ (1) means in minimal logic that C is a syntactical consequence of both A and B, ... In **IR** the meaning of (1) changes if either A or B is equivalent to $\bot$, because via [Tennant's version of] $\vee$E, $\bot \vee B \vdash B$ but only because of the axiom $B \vdash B$ and in spite of the claim that $\bot \nvdash B$ in **IR**. Therefore **IR** changes the meaning of $\vee$ when $\vee$ is on the left of the turnstile.

But, among other problems, why on earth should the observation that **IR** differs from minimal logic, and hence that a reading of the connectives which might be apt for the minimalist is not available to Tennant, be thought to be an *objection*? Tennant is just not committed to the thought that minimal logic already suffices to reflect the 'established meaning' of the connectives.

It seems that Vidal-Rosset is being led astray by a remark in 'Relevance in Reasoning' which he quotes out of context, where Tennant does indeed claim that his tweaking with the rules for the logical operators – e.g. in liberalizing of the form of proof by cases as above – "are not such as to change their established meanings". But context is all. The relevant paragraph starts "We relevantize both [classical logic] and [intuitionistic logic] by modifying the usual Tarskian assumptions about what is desired of a deducibility relation." So the local assumption is that you are starting as a conventional classical or intuitionist logician (not a minimalist!). Tennant's thought is that, either way, given your preferred story about the meaning of the logical operators, then you indeed should find no objection – nothing counter to your understanding as articulated in your preferred story – to the way that e.g. proof by cases is framed by Tennant. The departure from non-relevant classical/intuitionist logic comes elsewhere, in the rules for proof-building – the requirement, e.g., that a kosher proof be in normal form, with no detours where a wff is both the conclusion of an introduction rule and the major premiss of an elimination rule. This thought is entirely compatible with the claim that **IR** differs from minimal logic in such a way as not to tally with the naive semantic account of disjunction which Vidal-Rosset offers the minimal logician.

Later, in supposedly summarizing his argument here, Vidal-Rosset comes up with a different line:

> We expect that a Core logic respects the univocal meaning of logical rules. It is not the case with **IR**: Tennant's $\vee$E has not the same meaning when it is applied to a sequent like $A \vee \neg\neg\neg\neg A \vdash \neg\neg A$ (2) and when it is applied to $(\neg A \wedge A) \vee B \vdash B$ (3). Sequent (2) is valid in **IR** because two sequents are provable, i.e. $A \vdash \neg\neg A$ and $\neg\neg\neg\neg A \vdash \neg\neg A$, but sequent (3) is valid in **IR** only because $B \vdash B$ is an axiom.

But that's hopeless. For a start, you need still two subproofs in the proof of (3) too, one from $A \wedge \neg A$ to $\bot$ as well as the quite trivial proof from B to itself. True, you can frame Tennant's $\vee$E as a four-part rule (one part for each case, where the sub-proofs have respectively conclusions $\bot\bot$, $\bot C$, $C\bot$ or CC), and in different proofs different parts of $\vee$E will be invoked. So what? You wouldn't say that $\wedge$E doesn't respect the univocal meaning of logical rules because it has two parts, different parts being used in different proofs depending on whether you want to extract a left or right conjunct. So what's the difference? Vidal-Rosset doesn't say.

2 Irrelevance redux?

I'll next take Vidal-Rosset's third argument, in a simplified form. Any sequent of the following form is, of course, trivially provable in **IR**: $\phi \wedge (\phi \to \psi) \vdash \psi$. So, in particular, we have the instance

$$(A \wedge \neg A) \wedge ((A \wedge \neg A) \to B) \vdash B, \quad (4)$$

which is therefore provable in **IR** but also, intuitively, doesn't offend against any natural principle of relevance. No evident problem so far.

But now note that the second conjunct of the premiss here, i.e. $(A \wedge \neg A) \to B$ (5), is in fact a theorem of **IR**:

$$\cfrac{\cfrac{\cfrac{[A \wedge \neg A]}{A} \qquad \cfrac{[A \wedge \neg A]}{\neg A}}{\bot}}{(A \wedge \neg A) \to B}$$

Here, the final step invokes Tennant's tweaked version of conditional proof. Someone might worry whether this tweaked version, which allows us to pull a consequent out of thin air if an antecedent leads to contradiction, is sufficiently 'relevantist' in spirit. But for Tennant the *conditional* – in its idealized mathematician's usage – is *not* a relevant conditional: $(A \wedge \neg A) \to B$ can't fail (it is determinately ruled out that it should have a true antecedent and false consequent) so is assertible.

If we could glue the proof of this theorem (5) together with the proof of (4), we'd get

$$\begin{array}{c} \cfrac{A \wedge \neg A \qquad \begin{array}{c}\vdots \\ (A \wedge \neg A) \to B\end{array}}{(A \wedge \neg A) \wedge ((A \wedge \neg A) \to B)} \\ \vdots \\ B \end{array}$$

with no undischarged premisses at the top of the right hand subproof, establishing the unwanted irrelevancy $A \wedge \neg A \vdash B$. But unpacking further, we find that the proof here is not normalized or normalizable. The $\wedge$I as displayed will be immediately followed by applications of $\wedge$E to recover the conjuncts; and if we remove that detour, we are left with the application of $\to$I at the end of the (sub)proof of (5) followed an application of $\to$E. That's banned in **IR**. So we don't after all recover the unwanted ex falso inference.

So what's Vidal-Rosset's objection hereabouts? Allowing for our simplification of his example, he proves that (5) is a theorem, and then says this theorem

> leads to a way of avoiding in **IR** the relevance at the level of the turnstile: In **IR**, B is deducible "at the level of the turnstile" from the assumption of the conjunction of contradiction $A \wedge \neg A$ and the theorem $(A \wedge \neg A) \rightarrow B$, [i.e. we have (4)].

But what does Vidal-Rosset mean by saying that (5) "leads to a way of avoiding the relevance" given (4)? He doesn't explain. The thought might *seem* to be that since the second conjunct in the premiss of (4) is a theorem we can get it for free and recover ex falso. But we've just noted that that's simply wrong: we can't paste proofs together willy-nilly in **IR**, and in particular we can't do it in this case. And if Vidal-Rosset isn't falling into that trap, it is quite unclear what the argument is.

Later, in his summary of this supposed objection, Vidal-Rosset says something different and simpler:

> Why should be [Ex Falso Quodlibet, in the form $(\neg A \wedge A) \vdash B$] be unprovable, while the conditional $(\neg A \wedge A) \rightarrow B$ is a theorem of **IR**? This lack of harmony between provable sequents and provable conditionals is shocking.

But as we said before, Tennant's conditional is not relevant, while his turnstile in **IR** is. His conditional is, in a sense, governed by the usual truth-table: the difference between a classical and intuitionist reading of the table arises over the question whether it is to be supposed, as a matter of logic, that the world must always determinately fix (perhaps in a verification-transcendent way) which line we are living on. His turnstile, on the other hand, is governed by a deducibility relation governed by certain claimed canons of good reasoning. It is then, for Tennant, a *result* that the conditional and the turnstile are not (universally) in the harmony Vidal-Rosset presupposes. A shocked stare is not a counterargument.

3 Equivalents and substitution

Vidal-Rosset's remaining argument is clearer and more straightforward, but no better. According to him, the following are very basic facts about **IR**.

(a) $A \vdash A \vee \bot$ (trivially, by $\vee$I)

(b) $A \vee \bot \vdash A$ (trivially, by Tennant's version of $\vee$E)

(c) $\bot \vdash A \vee \bot$ (trivially, by $\vee$I again)

(d) $\bot \nvdash A$ (**IR** doesn't have Ex Falso).

Again, we can't paste together proofs of (c) followed by (b) to recover Ex Falso because that would produce an unnormalized tree with an application of $\vee$I immediately followed by an application of $\vee$E.

Now what this (supposedly) illustrates is that (S): we can have a pair of wffs ψ, ψ' which are equivalent in the strong sense that $\psi \dashv\vdash \psi'$, and also a proposition ϕ such that $\phi \vdash \psi$ but $\phi \nvdash \psi'$.

Now in fact, we should balk at Vidal-Rosset's supposed illustration of (S), since in **IR** $\perp$ serves to mark arriving at a contradiction (like the star coventionally used in tableaux proofs) and is not a wff that can occur as a subwff. But let that pass, as it is easy to construct examples using $C \wedge \neg C$ instead of $\perp$.

Vidal-Rosset initially just remarks on this phenomenon that (S) holds, i.e. "the law of substitution of logical equivalents fails in **IR**", as if it is obvious that this is a fatal problem. And later he writes "This law is essential to our understanding of the rules of logical systems in general." But why so? Compare: the "law" of the unrestricted transitivity of deducibility fails in **IR**. But it would be thumpingly question-begging to offer *this* observation as an objection, and say that this law is essential to logical systems, given that Tennant shows carefully how transitivity only fails in **IR** when – as he would say – it *ought* to fail, e.g. when bringing together the premisses of two different inferences gives us combined premisses which include an explicitly contradictory pair, so we should (so to speak) stop what we are are doing and revise our premisses rather than blithely carry on. Since the failure of the "law" of substitution is an immediate consequence of the failure of transitivity, why is insisting on the universal intersubstitutibility of interderivable wffs any less question-begging?

Vidal-Rosset writes later (changed to avoid an occurrence of $\perp$ as a subwff)

> The trouble is that the logical equivalence between $A \vee (C \wedge \neg C)$ and A is syntactically provable in **IR** regardless of context, and therefore should be semantically valid in **IR** also regardless of context; proof has been given in this paper that it is not the case in **IR**, and it is a serious problem if harmony between syntax and semantics is expected in Core logic.

But Tennant would not deny that if $A \vee (C \wedge \neg C)$ is true, so is the interderivable A and vice versa (and nothing in Vidal-Rosset's paper proves otherwise). Tennant's position, rather, is that in order for a properly constructed deduction to be available – one governed by the appropriate syntactic constraints of relevance – it can matter which of these two semantically equivalent wffs features. Again, it needs an argument, not an incredulous stare, to establish that *this* fundamental feature of **IR** rules it out of consideration as a "core logic".

University of Cambridge