Squeezing arguments

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Many of our concepts are introduced to us via, and seem only to be constrained by, rough-and-ready explanations and some sample paradigm positive and negative applications. This happens even in informal logic and mathematics. Yet in *some* cases, the concepts in question – although only informally and vaguely characterized – in fact have, or appear to have, entirely determinate extensions.

Here's one familiar example. When we start learning computability theory, we are introduced to the idea of an *algorithmically computable function* (from numbers to numbers) – i.e. one whose value for any given input can be determined by a step-by-step calculating procedure, where each step is fully determined by some antecedently given finite set of calculating rules. We are told that we are to abstract from practical considerations of how many steps will be needed and how much ink will be spilt in the process, so long as everything remains finite. We are also told that each step is to be 'small' and the rules governing it must be 'simple', available to a cognitively limited calculating agent: for we want an algorithmic procedure, step-by-minimal-step, to be idiot-proof. For a classic elucidation of this kind, see e.g. Rogers (1967, pp. 1–5). Church's Thesis, in one form, then claims this informally explicated concept in fact has a perfectly precise extension, the set of recursive functions.

Church's Thesis can be supported in a quasi-empirical way, by the failure of our searches for counterexamples. It can be supported too in a more principled way, by the observation that different appealing ways of sharpening up the informal chararacterization of algorithmic computability end up specifying the same set of recursive functions. But such considerations fall short of a *demonstration* of the Thesis. So is there a different argumentative strategy we could use, one that could lead to a proof?

Sometimes it is claimed that there just can't be, because you can never really *prove* results involving an informal concept like algorithmic computability. But absolutely not so. Consider, for just one example, the diagonal argument that shows there are algorithmically computable functions that are not primitive recursive. That's a mathematical proof by any sane standard, and its conclusion is quite rightly labelled a theorem in standard textbooks.

So our question remains. To generalize it: can there be a strategy for showing of an informally characterized concept that it does indeed have the same extension as some sharply defined concept?

1 Squeezing arguments, the very idea

Here, outlined in very schematic form, is one type of argument that *would* deliver such a co-extensiveness result.

Take a given informally characterized concept I. And suppose firstly that we can find some precisely defined concept S such that – in the light of that characterization

– falling under concept S is certainly and uncontroversially a *sufficient* condition for falling under the concept I. So, when e is some entity of the appropriate kind for the predications to make sense, we have

K1. If e is S, then e is I.

Now suppose secondly that we can find another precisely defined concept N such that falling under concept N is similarly an uncontroversial *necessary* condition for falling under the concept I. Then we also have

K2. If e is I, then e is N.

In terms of extensions, therefore, we have

Ki. $|S| \subseteq |I| \subseteq |N|$

where $|X|$ is the extension of X. So the extension of I – vaguely gestured at and indeterminately bounded though that might be – is at least sandwiched between the determinately bounded extensions of S and N.

So far, so uninteresting. It is no news at all that even the possibly fuzzy extensions of paradigmatically vague concepts can be sandwiched between those of more sharply bounded concepts. The extension of 'tall' (as applied to men) is sandwiched between those of 'over five foot' and 'over seven foot'.

But now suppose, just suppose, that in a particular case our informal concept I gets sandwiched between such sharply defined concepts S and N, but we can *also* show that

K3. If e is N, then e is S.

In the sort of cases we are going to be interested in, I will be an informal logical or mathematical concept, and S and N will be precisely defined concepts from some rigorous theory. So in principle, the possibility is on the cards that the result K3 could actually be a *theorem* of the relevant mathematical theory. But in that case, we'd have

Kii. $|S| \subseteq |I| \subseteq |N| \subseteq |S|$

so the inclusions can't be proper.

What's happened is that the theorem K3 squeezes together the extensions $|S|$ and $|N|$ which are sandwiching the extension I, and we have to conclude

Kiii. $|S| = |I| = |N|$

In sum, the extension of the informally characterized concept I is now revealed to be just the same as the extensions of the sharply circumscribed concepts S and N.

All this, however, is merely schematic. The next and crucial question is: are there any plausible cases of informal concepts I where this sort of squeezing argument *can* be mounted, and we *can* show in this way that the extension of I is indeed the same as that of some sharply defined concept?

2 Kreisel's squeezing argument

Well, there's certainly one persuasive candidate example – due to Georg Kreisel (1972), to whom the general idea of such a squeezing argument is ultimately due. But the example seems much less familiar than once it was. And to understand the general prospects for squeezing arguments, it is important to get his argument back into clear focus, to understand what it does and doesn't establish. The most recent discussion of it badly misses the mark.

So, take the entities being talked about to be *arguments couched in a given regimented first-order syntax with a standard semantics.* Here we mean of course arguments whose language has the usual truth-functional connectives, and whose quantifiers are understood classically (in effect, as potentially infinitary conjunctions and disjunctions). And now consider the concept I_L, the informal notion of *being valid-in-virtue-of-form* for such arguments.

As a first shot, we informally elucidate this concept by saying that an argument α is valid in this sense if, however we spin the interpretations of the non-logical vocabulary, and however we pretend the world is, it's never the case that α's premisses come out true and its conclusion false. Then, noting that, on the standard semantics for a first-order language, everything is extensional, we can – as a second shot – put the idea like this: α is valid just if, whatever things we take the world to contain, whichever of those things we re-interpret names to refer to, and whatever extensions among those things we re-interpret predicates as picking out, it remains the case that whenever α's premisses come out true, so does its conclusion.

Of course, that explication takes us some distance from a merely 'intuitive' notion of validity (if such there be – more about that in the next section). But it is still vague and informal: it's the sort of loose explanation we give in an introductory logic course. In particular, we've said nothing explicitly about where we can look for the 'things' to build those structures of objects and extensions which the account of validity generalizes over. For example, just how big a set-theoretic universe can we call on? – which of your local mathematician's tall stories about wildly proliferating hierarchies of 'objects' do you actually take seriously enough to treat as potential sources of structures that we need to care about? If you do cheerfully buy into set-theory, what about allowing domains of objects that are even bigger than set-sized? Our informal explication just doesn't speak to such questions.

But no matter; informal though the explication is, it does in fact suffice to pin down a unique extension for I_L. Here's how.

Take S_L to be the property of having a proof for your favourite natural-deduction proof system for classical first-order logic. Then (for any argument α)

L1. If α is S_L, then α is I_L.

That is to say, the proof system is classically sound: if you can formally deduce φ from some bunch of premisses Σ, then the inference from Σ to φ is valid according to the elucidated conception of validity-in-virtue-of-form. That follows by an induction on the length of the proofs, given that the basic rules of inference are sound according to our conception of validity, and chaining inference steps preserves validity. Their validity in that sense is, after all, the principal reason why classical logicians accept the proof system's rules in the first place!

Second, let's take N_L to be the property of *having no countermodel in the natural numbers.* A countermodel for an argument is, of course, an interpretation that makes the premisses true and conclusion false; and a countermodel in the natural numbers is one whose domain of quantification is the natural numbers, where any constants refer to numbers, predicates have sets of numbers as their extensions, and so forth. Now, even if we are more than a bit foggy about the limits to what counts as legitimate re-interpretations of names and predicates as mentioned in our informal explication of the idea of validity, we must surely recognize at least this much: if an argument does have a countermodel in the natural numbers – i.e. if we can reconstrue the argument to be talking about natural numbers in such a way that actually makes the premisses are true and conclusion false – then the argument certainly can't be valid-in-virtue-of-its-form in the informal sense. Contraposing,

L2. If α is I_L, then α is N_L.

So the intuitive notion of validity-in-virtue-of-form (for inferences in our first-order language) is sandwiched between the notions of being provable in your favourite system, and having no arithmetical counter-model, and we have

Li. $|S_L| \subseteq |I_L| \subseteq |N_L|$

But now, of course, it's a standard theorem that

L3. If α is N_L, then α is S_L.

That is to say, if α has no countermodel in the natural numbers, then α can be deductively warranted in your favourite classical natural deduction system. That's just a corollary of the usual proof of the completeness theorem for first-order logic.

So L3 squeezes the sandwich together. We can conclude, therefore, that

Liii. $|S_L| = |I_L| = |N_L|$

In sum, take the relatively informal notion I_L of a first-order inference which is valid in virtue of its form (explicated as sketched): then our pre-theoretic assumptions about that notion constrain it to be coextensive with each of two sharply defined, mutually coextensive, formal concepts.

3 Contra Field: what Kreisel's argument doesn't show

Now, let's not get overexcited! We *haven't* magically shown, by waving a techno-flash wand, that an argument (in a first-order language) is 'intuitively valid' if and only if it is valid on the usual post-Tarski definition.

Recently, however, Hartry Field (2008, pp. 47–48) has presented Kreisel squeezing argument as having the magical conclusion. Field explicitly takes the concept featuring in the squeeze to be 'the intuitive notion of validity'; and he says the conclusion of Kreisel's argument is that we can 'use intuitive principles about validity, together with technical results from model theory, to argue that validity [meaning the intuitive notion] extensionally coincides with the technical [model-theoretic] notion'. But Field is wrong, both in his representation of Kreisel's own position, and about what a Kreisel-style argument might hope to establish.

Now, it is true that Kreisel initially defines the informal concept *Val* that features in his own argument by saying that '*Val* α' means 'α is intuitively valid'. But then Kreisel immediately goes on to explicate *that* as saying that α is 'true in all structures' (note then that he is in fact squeezing on a notion of validity for propositions rather than for arguments – but we'll not worry about this, for it doesn't effect the issue at stake). And although he doesn't say a great deal more about the idea of truth in a structure, it is clear enough that for him structures are what we get by picking a universe of objects (to be the domain of quantification) and then assigning appropriate extensions from this universe to names and predicates. In other words, Kreisel's notion of validity is the analogue for propositions of our explicated notion I_L of validity for arguments. So, for him, it *isn't* some raw 'intuitive' notion of validity that at stake: rather it is a more refined idea that has already been subject to an amount of sharpening, albeit of an informal sort.

And that's surely necessary if the squeezing argument is to have any hope of success. For there just is no pre-theoretical 'intuitive' notion of valid consequence with enough shape to it for such an argument to get a grip.

If you think that there is, start asking yourself questions like this. Is the intuitive notion of consequence constrained by considerations of relevance? – do ex falso quodlibet inferences commit a fallacy of relevance? When can you suppress necessarily true premisses and still have an inference which is intuitively valid? What about the inference 'The cup contains some water; so it contains some H_2O molecules'? That necessarily preserves truth (on Kripkean assumptions): but is it valid in the intuitive sense? – if not, just why not?

Such questions surely lack determinate answers: we can be pulled in various directions. I'm entirely with Timothy Smiley (1988) when he remarks that the idea of a valid consequence is 'an idea that comes with a history attached to it, and those who blithely appeal to an "intuitive" or "pre-theoretic" idea of consequence are likely to have got hold of just one strand in a string of diverse theories.' For more elaboration, see Smiley's article.

Contra Field, then, there seems no hope for a squeezing argument to show that our initial inchoate, shifting, intuitions about validity – such as they are – succeed in pinning down a unique extension (at least among arguments cast in a first-order vocabulary). You can't magically wave away relevantist concerns, for example. And Kreisel himself doesn't claim otherwise.

4 What Kreisel's argument does show

The idea, then, is better seen as follows. *One* way of beginning to sharpen up our inchoate intuitive ideas about validity – still informal, but pushing us in certain directions with respect to those questions we've just raised – is this. We say that an inference is valid in virtue of form if there's no case which respects the meaning of the logical constants where the premisses are true and conclusion false. That already warrants ex falso as a limiting case of a valid argument. And given that 'water' and 'H_2O' are bits of non-logical vocabulary, that means that the inference 'The cup contains water; so it contains H_2O' is of course *not* valid in virtue of form.

But now we need to say more about what 'cases' are. After all, an intuitionist might here start talking about 'cases' in terms of warrants or constructions. Pushing things in a classical direction, we start to elucidate talk about cases in terms of ways-of-making-true: an inference is valid-in-virtue-of-form when if, whatever we take the relevant non-logical vocabulary to mean, and however the world turns out, it can't be that α's premisses are true and its conclusion is false. Then, given we are talking about a first-order language where it is extensions that do the work of fixing truth-values, we further explicate this idea along Kreisel's lines: argument validity is a matter of there being no structure – no universe and assignment of extensions – which makes the premisses true and conclusion false.

And it is only *now* that Kreisel's squeezing argument kicks in. It shows that, having done *this* much informal tidying, although on the face of it we've still left things rather vague and unspecific, *in fact* we've done enough to fix a determinate extension for the notion of validity-in-virtue-of-form (at least as applied to arguments cast in a first-order vocabulary).

Put it like this. There are three conceptual levels here:

1. We start with a rather inchoate jumble of ideas of validity (as Smiley suggests, there is no single 'intuitive' concept here).

2. We can sort things out in various directions. Pushing some way along in one direction (and there are other ways we could go, equally well rooted – ask any relevantist!), we get an informal, still somewhat rough-and-ready classical notion of validity-in-virtue-of-form.

3. Then there are crisply defined notions like derivability-in-your-favourite-deductive system and the modern post-Tarski notion of validity.

The move from the first to the second level involves a certain exercise in conceptual sharpening. And there is no doubt a very interesting story to be told about the conceptual dynamics involved such a reduction in the amount of 'open-texture', as we get rid of some of the imprecision in our initial inchoate ideas and privilege some strands over other – for this exercise isn't an *arbitrary* one. However, it plainly would be over-ambitious to claim that in refining our inchoate ideas and homing in on the idea of validity-in-virtue-of-form (explicated in terms of preserving truth over all structures) we are just explaining what we were talking about all along. There's too much slack in our initial ideas; we can develop them in different directions. And it is only after we've got to the second level that the squeezing argument bites: the claim is that – less ambitiously but still perhaps surprisingly – we don't have to sharpen things completely before (so to speak) the narrowing extension of validity snaps into place and we fix on the extension of the modern post-Tarski notion.

5 The prospects for squeezing arguments

And that, I suggest, is going to be typical of other potential squeezing arguments.

To return to our initial example, what are the prospects of running a squeezing argument on the notion of a computable function? None at all. Or at least, none at all if we really do mean to start from a very inchoate notion of computability guided just by some initial vague explanations and a sample of paradigms. Ask yourself: before our ideas are too touched by theorizing, what kind of 'can' is involved in the idea of a function that 'can be computed'? Can be computed by us, by machines? By us (or machines) as in fact we or as we could be? Constrained by what laws, the laws as they are or as they could be in some near enough possible world? Is the idea of computability tied to ideas of feasibility at all? I take that such questions have no determinate answers any more than the comparable questions we had about a supposed intuitive notion of validity.

As with the notion of validity, if we are going to do any serious work with a notion of computability, we need to start sharpening up our ideas. And as with the notion of validity, there are various ways to go. *One* familiar line of development takes us to the sharper though still informal notion of a finite *algorithmic symbolic computation*. But there are other ways to go (ask any enthusiast for the coherence of ideas of hypercomputation).

So here too there are three levels of concepts which can be in play hereabouts:

1. We start with initial, inchoate, 'unrefined', ideas of *computability* – ideas which are fixed, insofar as they *are* fixed, by reference to some paradigms of common-or-

garden real-world computation, and perhaps some arm-waving explanations (like 'what some machine might compute').

2. Next there is our idealized though still informal and vaguely framed notion of computability using a symbolic algorithm (also, of course, known as 'effective computability').

3. Then, thirdly, there are the formal concepts such as *recursiveness* and *Turing computability* (and concepts of hypercomputation and so on with different extensions).

And again, it would plainly be over-ambitious to claim that in refining our inchoate ideas and homing in on the idea of effective computability we are just explaining what we were talking about all along. There's again too much slack in our initial position. Rather, Church's Thesis – or at least the version that most interests me (and, I would argue, the founding fathers too) – kicks in at the *next* stage. The claim is that, once we have arrived at the second, more refined but still somewhat vague, concept of an algorithmic computable function, *then* we've got a concept which has as its extension just the same unique class of functions as the third-level concepts of recursive or Turing-computable functions. And it is here, if anywhere, that we might again try to bring to bear a squeezing argument.

Now, as it happens, I think such an argument is available (see Smith, 2007, ch. 35): but it isn't my present concern to make that case. Rather, what I've tried to do in this note is to make it much clearer what role which Kreisel's original squeezing argument has – not, pace Field, in fixing the extension of an 'intuitive' concept, but in fixing the extension of informally characterized but semi-technical idea. It will be similar, I claim, with other plausible candidate squeezing arguments. So understood, the ambitions of squeezing arguments are less radical than on a Fieldian reading: but the chances of some successes are much higher.

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