Mathematical Thought and Its Objects

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Needless to say, Charles Parsons's long awaited book[1] is a must-read for anyone with an interest in the philosophy of mathematics. But as Parsons himself says, this has been a very long time in the writing. Its chapters extensively "draw on", "incorporate material from", "overlap considerably with", or "are expanded versions of" papers published over the last twenty-five or so years. What we are reading is thus a multi-layered text with different passages added at different times. And this makes for a rather bumpy read. There is another route Parsons could have taken: he could have reprinted the relevant papers with postscripts, and then top-and-tailed the collection with a preface and added concluding reflections. It must sound very ungrateful, but I rather suspect that that might have worked better.

Much of the book is about arithmetic. But Parsons has woven into the discussion claims about mathematics more generally and about set theory in particular. We might well have a basic worry about this structure: for do defensible claims about the ontology and epistemology of arithmetic have to be generalizable to apply to more infinitary mathematics? For example, suppose you are attracted to a Hellman-like modal structuralist account of arithmetic: then should you think it a problem if you suspect that such an account can't readily be extended to cope e.g. with set theory (since you boggle at the idea of possible worlds free of abstracta but with enough structure to somehow model ZFC)? Parsons himself seems to waver over such questions. So for present purposes, I'll focus just on arithmetic, and in what follows I'll revisit two of the most familiar Parsonian themes, his views on *structuralism* as an account of the ontology of arithmetic, and his exploration of the role of *intuition* in grounding arithmetical knowledge.

Noneliminative structuralism In his first chapter, Parsons defends a thin, logical, conception of 'objects' on which "Speaking of objects just is using the linguistic devices of singular terms, predication, identity and quantication to make serious statements" (p. 10). His second chapter critically discusses eliminative structuralism. The third chapter presses objections against modal structuralism. But Parsons still finds himself wanting to say that "something close to the structuralist view is true" (p. 42), and he now moves on characterize his own preferred *noneliminative* version.

Parsons makes two key initial points. (1) Unlike the eliminative structuralist, the noneliminativist "take[s] the language of mathematics at face value" (p. 100). So arithmetic is indeed about *numbers* as objects. What characterizes the position as structuralist is that we don't "take more as objectively determined about the objects about which it speaks than [the relevant mathematical] language itself specifies" (p. 100). (2) Then there is "the aspect of the structuralist view stressed by Bernays, that existence for mathematical objects is in the context of a background structure" (p. 101). Further, structures aren't themselves objects, and "[the noneliminativist] structuralist account of a particular kind of mathematical object does not view statements about that kind of object as about structures at all".

But note, thus far there's nothing in (1) and (2) that e.g. the neo-Fregean Platonist need dissent from. The neo-Fregean can agree e.g. that numbers only have numerical

[1] Mathematical Thought and Its Objects by Charles Parsons (Cambridge University Press, 2008). xx + 378 pp. £50. You can usefully inspect the detailed table of contents at the publisher's website: http://www.cambridge.org

intrinsic properties (pace Frege himself, even raising the Julius Caesar problem is a kind of mistake). Moreover, he can insist that individual numbers don't come (so to speak) separately, one at a time, but come all together forming an intrinsically order structured – so in, identifying the number 42 as such, we necessarily give its position in relation to other numbers.[2]

So what *more* is Parsons saying about numbers that distinguishes his position from the neo-Fregean? Well, he in fact explicitly compares his favoured structuralism with the view that the natural numbers are *sui generis* in the sort of way that the neo-Fregean holds. He writes

> One further step that the structuralist view takes is to reject the demand for any further story about what objects the natural numbers are [or are not]. (p. 101)

The picture seems to be that the neo-Fregean offers a "further story" at least in negatively insisting that numbers are sui generis, while the structuralist refuses to give such a story. As Parsons puts it elsewhere

> If what the numbers are is determined only by the structure of numbers, it should not be part of the nature of numbers that none of them is identical to an object given independently.[3]

But of course, neo-Fregeans like Hale and Wright won't agree that their rejection of cross-type identities is somehow an optional extra: they offer arguments which – successfully or otherwise – aim to block the Julius Caesar problem and reveal certain questions about cross-type identifications as ruled out by our very grasp of the content of number talk. So from this neo-Fregean perspective, we can't just wish into existence a coherent structuralist position that both (a) construes our arithmetical talk at face value, as referring to numbers as genuine objects, yet also (b) insists that the possibility of cross-type identifications is left open, because – so the story goes – a properly worked out version of (a), together with reflection on the ways that genuine objects are identified under sortals, implies that we can't allow (b).

Now, on the sui generis view about numbers, claims identifying numbers with sets will be ruled out as plain false. Or perhaps it is even worse, and such claims fail to make the grade for being either true or false (though it is, of course, notoriously difficult to sustain a stable, well-motivated, distinction between the neither-true-nor-false and the plain false – so let's not dwell on this). Conversely, assuming that numbers are objects, if claims identifying them with sets and the like are false or worse, then numbers are sui generis. So it seems that if Parsons is going to say that numbers *are* objects but are *not* sui generis, he must allow that claims identifying numbers with sets (or if not sets, at least some other objects) might be true. But then Parsons is faced with the familiar Benacerraf multiple-candidates problem (if not for sets, then presumably an analogous problem for other candidate objects, whatever they are: let's keep things simple by running the argument in the familiar set-theoretic setting). How *do* we choose e.g. between saying that the finite von Neumann ordinals are the natural numbers and saying that the finite Zermelo ordinals are?

It seems arbitrary to plump for either choice. Rejecting both together (and other choices, on similar grounds) just takes us back to the sui generis view – or even to

[2] Hale and Wright say as much in their *The Reason's Proper Study*, p. 1, fn. 1.

[3] C. Parsons, 'Structuralism and metaphysics', *Philosophical Quarterly* 2004, p. 61.

Benacerraf's preferred view that numbers aren't objects at all. So that, it seems, leaves just one position open to Parsons, namely to embrace *both* choices, and to avoid the apparently inevitable absurdity that $\{\emptyset, \{\emptyset\}\}$ is identical to $\{\{\emptyset\}\}$ (because both are identical to 2) by going contextual. It's only in one context that '$2 = \{\emptyset, \{\emptyset\}\}$' is true; and only in another that '$2 = \{\{\emptyset\}\}$' is true.

And this does seem to be the line Parsons seems inclined to take: "The view we have defended implies that [numbers] are not definite objects, in that the reference of terms such as 'the natural number two' is not invariant over all contexts" (p. 106). But how are we to understand that? Is it supposed to be rather like the case where, when Brummidge United is salient, 'the goal keeper' refers to Joe Doe, but when Smoketown City is salient, 'the goal keeper' refers to Richard Roe? So when the von Neumann ordinals are salient, '2' refers to $\{\emptyset, \{\emptyset\}\}$ and the Zermelo ordinals are salient, '2' refers to $\{\{\emptyset\}\}$? But then, to pursue the analogy, while 'the goal keeper' is indeed sometimes used to talk about now this particular role-filler and now that one, the designator is apparently also sometimes used more abstractly to talk about the role itself – as when we say that only the goal keeper is allowed to handle the ball. Likewise, even if we do grant that '2' sometimes refers to role-fillers, it seems that sometimes it is used to talk more abstractly about the role – perhaps as when we say, when no particular ω-sequence of sets is salient, that 2 is the successor of the successor of zero. Well, is *this* the way Parsons is inclined to go, i.e. towards a structuralism developed in terms of a metaphysics of roles and role-fillers?

Well, Parsons does explicitly talk of "the conclusion that natural numbers are in the end roles rather than objects with a definite identity" (p. 105). *But why aren't roles objects after all, in his official thin 'logical' sense?* – for we can use "the linguistic devices of singular terms, predication, identity and quantification to make serious statements" about roles (and yes, we surely *can* make claims about identity and non-identity: the goal keeper is not the striker). True, roles are as Parsons might say, "thin" or "impoverished" objects whose intrinsic properties are determined by their place in a structure. But note, Parsons's official view about objects didn't require any sort of 'thickness': indeed, he is "most concerned to reject the idea that we don't have genuine reference to objects if the 'objects' are impoverished in the way in which elements of mathematical structures appear to be" (p. 107). And being merely 'thin' objects, roles themselves (e.g. numbers) can't be the same things as 'thick' role-fillers. So now, after all, numbers qua number-roles do look to be sui generis entities with their own identity – objects, in the broad logical sense, which are not to be identified with any role-filler – in other words, just the kind of thing that Parsons seems *not* to want to be committed to.

The situation is further complicated when Parsons briefly discusses Dedekind abstraction, though similar issues arise. To explain: suppose we have a variety of 'concrete' structures, whether physically realized or realized in the universe of sets, that satisfy the conditions for being a simply infinite system. Then Dedekind's idea is that we 'abstract' from these a *further* structure $\langle N, 0, S\rangle$ which is – so to speak – a 'bare' simply infinite system without other inessential physical or set-theoretic features, and it is elements of *this* system which are the numbers themselves. (Erich H. Reck nicely puts it like this:[4] "[W]hat is the system of natural numbers now? It is that simple infinity whose objects only have arithmetic properties, not any of the additional, 'foreign' properties objects in other simple infinities have.") Since the bare structure is *all* that is generated by the Dedekind abstraction, "it conforms to the basic structuralist intuition in that the number terms introduced do not give us more than the structure" (p. 105), to borrow

[4] In his very illuminating article on 'Dedekind's structuralism', *Synthese* 2003, p. 400.

Parsons's words. But, he continues,

> This procedure gets its force from the use of a typed language. Thus, the question arises what is to prevent us from later, for some specific purpose, speaking of numbers in a first-order language and even affirming identities of numbers and objects given otherwise.

To which the answer surely is that, to repeat, on the Dedekind abstraction view, the 'thin' numbers determinately *don't* have intrinsic properties other than those given in the abstraction procedure which introduces them: so, by assumption, they are determinately distinct from any 'thicker' object with such further properties. Why not?

So now I'm puzzled. For Parsons, does 'the natural number two' (i) have a fixed reference to a sui-generis 'thin' role-object (or Dedekind abstraction, if that's different), or (ii) have a contextually shifting reference to a role-filler, or (iii) both? The latter is perhaps the most charitable reading. But it would have helped a lot if Parsons had much more explicitly related his position to an articulated metaphysics of role/role-filler structuralism. Elsewhere, he writes that "the metaphysical tradition is likely to be misleading as a source of ideas about the objects of modern mathematics".[5] Maybe that's right. But then it is all the more important to be absolutely clear and explicit about what new view is being proposed.

Intuition Is any of our arithmetical knowledge *intuitive* knowledge, grounded on intuitions of mathematical objects? Parsons writes, "It is hard to see what could make a cognitive relation to objects count as intuition if not some analogy with perception" (p. 144). But how is such an analogy to be developed?

Parsons tries to soften us up for the idea that we can have intuitions of abstracta (where these intuitions are somehow quasi-perceptual) by considering the putative case of perceptions – or are they intuitions? – of abstract types such as letters. The claim is that "the talk of perception of types is something normal and everyday" (p. 159).

But it is of course not enough to remark that we *talk* of e.g. seeing types: we need to argue that we can take our talk here as indeed reporting a (quasi)-perceptual relation to types. Well, here I am, looking at a squiggle on paper: I immediately see it as being, for example, a Greek letter phi. And we might well say: I see the letter phi written here. But, in this case, it might well be said, 'perception of the type' is surely a matter of perceiving the squiggle *as* a token of the type, i.e. perceiving the squiggle and taking it as a phi.

Now, it would be wrong to suppose that – at an experiential level – 'seeing as' just factors into a perception and the superadded exercise of a concept or of a recognitional ability. When the aspect changes, and I see the lines as a duck rather than a rabbit, at some level the content of my conscious perception itself, the way it is articulated, changes. Still, in seeing the lines as a duck, it isn't that there is *more* epistemic input than is given by sight (visual engagement with a humdrum object, the lines) together with the exercise of a concept or of a recognitional ability. Similarly, seeing the squiggle as a token of the Greek letter phi again doesn't require me to have some epistemic source over and above ordinary sight and conceptual/recognitional abilities. There is no need, it seems, to postulate something *further* going on, i.e. quasi-perceptual 'intuition' of the type.

[5]This is from 'Structuralism and Metaphysics' again, p. 56.

The deflationist idea, then, is that seeing the type phi instantiated on the page is a matter of seeing the written squiggle as a phi, and this involves bring to bear the concept of an instance of phi. And, the suggestion continues, having such a concept is *not* to be explained in terms of a quasi-perceptual cognitive relation with an abstract object, the type. If anything it goes the other way about: 'intuitive knowledge of types' is to be explained in terms of our conceptual capacities, and is not a further epistemic source. (But note, the deflationist who resists the stronger idea of intuition as a distinctive epistemic source isn't barred from taking Parsons's permissive line on objects, and can still allow the introduction of talk via abstraction principles of abstract objects such as types. He needn't have a nominalist horror of talk of abstracta.)

Let's be clear here. It may well be that, as a matter of the workings of our cognitive psychology, we recognize a squiggle as a token phi by comparing it with some stored template. But that of course does not imply that we need be able, at the personal level, to bring the template to consciousness: and even if we were to have some quasi-perceptual access to the *template* itself, it wouldn't follow that we have quasi-perceptual access to the *type*. Templates are mental representations, not the abstracta represented.

Parsons, however, explicitly rejects the sketched deflationary story about our intuition of types when he turns to consider the particular case of the perception of expressions from a very simple 'language', containing just one primitive symbol '|' (call it 'stroke'), which can be concatenated. The deflationary reading

> does not accurately render our perceptual consciousness of strokes. It would make what I want to call intuition of a string an instance of seeing a certain inscription *as* of a type .... But in actual cases, the identification of the type will be firmer and more explicit that the identification of any physical inscriptions that is an instance of the type. That the inscriptions are real physical objects with definite physical properties plays no role in the mathematical treatment of the language, which is what concerns us. An illusory presentation of a string, provided it is sufficiently clear, will do as well to illustrate a mathematical notion as a real one. (p. 161)

There seem to be two points here, neither of which will really trouble the deflationist.

The first point is that the identification of a squiggle's type may be "firmer and more explicit" than our determination of its physical properties as a token (which I suppose means that a somewhat blurry shape may still definitely be a letter phi). But so what? Suppose we have some discrete conceptual pigeon-holes, and have reason to take what we see as belonging in one pigeonhole or another (as when we are reading Greek script, primed with the thought that what we are seeing will be a sequence of letters from forty eight upper and lower case possibilities). Then fuzzy tokens can get sharply pigeonholed. But there's nothing here that the deflationist about seeing types can't accommodate.

The second point is that, for certain illustrative purposes, illusory strings are as good as physical strings. But again, so what? Why shouldn't seeing an illusory strokes as a string be a matter of our tricked perceptual apparatus engaging with our conceptual/recognitional abilities? Again, there is certainly no need to postulate some further cognitive achievement, 'intuition of a type'.

Oddly, Parsons himself, when wrestling with issues about vagueness, comes close to making these very points. For you might initially worry that intuitions which are founded in perceptions and imaginings will inherit the vagueness of those perceptions or imaginings – and how would that then square the idea that mathematical intuition latches onto sharply delineated objects? But Parsons moves to block the worry, using the example of seeing letters again. His thought now seems to be the one above, that we have some discrete conceptual pigeon-holes, and in seeing squiggles as a phi or a psi (say), we are pigeon-holing them. And the fact that some squiggles might be borderline candidates for putting in this or that pigeon-hole doesn't (so to speak) make the pigeon-holes less sharply delineated. Well, fair enough. But thinking in these terms surely does not sustain the idea that we need some basic notion of the intuition of the type phi to explain our pigeon-holing capacities.

So, I'm unpersuaded that we actually need (or indeed can make much sense of) any notion of the quasi-perceptual 'intuition of types' – and in particular, any notion of the intuition of types of stroke-strings – that resists a deflationary reading. But let's suppose for a moment that we follow Parsons and think we *can* make sense of such a notion. Then what use does he want to make of the idea of intuiting strokes and stroke-strings?

Parsons writes

> What is distinctive of intuitions of types [here, types of stroke-strings] is that the perceptions and imaginings that found them play a paradigmatic role. It is through this that intuition of a type can give rise to propositional knowledge about the type, an instance of intuition that. I will in these cases use the term 'intuitive knowledge'. A simple case is singular propositions about types, such as that ||| is the successor of ||. We see this to be true on the basis of a single intuition, but of course in its implications for tokens it is a general proposition. (p. 162)

This passage raises a couple of issues worth commenting on.

One issue concerns the claim that there is a 'single intuition' here on basis of which we see that that ||| is the successor of ||. I can think of a few cognitive situations which we might agree to describe as grounding quasi-perceptual knowledge that ||| is the successor of || (even if some of us would want to give a deflationary construal of the cases, one which doesn't appeal to intuition of abstracta). For example,

1. We perceive two stroke-strings

   ||
|||

   and aligning the two, we judge one to be the successor or the other.

2. We perceive a single sequence of three strokes ||| and flip to and fro between seeing it as a threesome and as a block of two followed by an extra stroke.

But, even going along with Parsons on intuition, neither of those cases seems aptly described as seeing something to be true on the basis of a *single* intuition. In the first case, don't we have an intuition of ||| and a separate intuition of || plus a recognition of the relation between them? In the second case, don't we have successive intuitions, and again a recognition of the relation between them? It seems that our knowledge that ||| is the successor of || is in either case grounded on intuitions, plural, plus a judgement about their relation. And now the suspicion is that it is the *thoughts* about the relations that really do the essential grounding of knowledge here (thoughts that could as well be engaging with perceived real tokens or with imagined tokens, rather than with putative Parsonian intuitions that, as it were, reach past the real or imagined inscriptions to the abstracta).

The other issue raised by the quoted passage concerns the way that the notion of 'intuitive knowledge' is introduced here, as the notion of propositional knowledge that arises in a very direct and non-inferential way from intuition(s) of the objects the knowledge is about: "an item of intuitive knowledge would be something that can be 'seen' to be true on the basis of intuiting objects that it is about" (p. 171). Such a notion looks very restrictive – on the face of it, there won't be much intuitive knowledge to be had.

But Parsons later wants to extend the notion in two ways. First

> Evidently, at least some simple, general propositions about strings can be seen to be true. I will argue that in at least some important cases of this kind, the correct description involves imagining *arbitrary* strings. Thus, that will be included in 'intuiting objects that a proposition is about'. (p. 171)

But even if we now allow intuition of 'arbitrary objects', that still would seem to leave intuitive knowledge essentially non-inferential. However,

> I do not wish to argue that the term 'intuitive knowledge' should not be used in that [restrictive] way. Our sense, following that of the Hilbert School, is a more extended one that allows that certain inferences preserve intuitive knowledge, so that there can actually be a developed body of mathematics that counts as intuitively known. This seems to me a more interesting conception, in addition to its historical significance. Once one has adopted this conception, one has to consider case by case what inferences preserve intuitive knowledge. (p. 172)

Two comments about his. Take the second proposed extension first. The obvious question to ask is: *what will constrain our case-by-case considerations of which kinds of inference preserve intuitive knowledge*? To repeat, the concept of intuitive knowledge was introduced by reference to an example of knowledge seemingly non-inferentially obtained. So how are we supposed to 'carry on', applying the concept now to inferential cases? It seems that nothing in our original way of introducing the concept tells us which such further applications are legitimate, and which aren't. But there must be *some* constraints here if our case-by-case examinations are not just to involve arbitrary decisions. So what are these constraints? I struggle to find any clear explanation in Parsons.

And what about intuiting 'arbitrary' strings? How does this ground, for example, the knowledge that every string has a successor? Well, supposedly, (1) "If we imagine any [particular] string of strokes, it is immediately apparent that a new stroke can be added." (p. 173) (2) But we can "leave inexplicit its articulation into single strokes" (p. 173), so we are imagining an arbitrary string, and it is evident that a new stroke can be added to this too. (3) "However, . . . it is clear that the kind of thought experiments I have been describing can be taken as intuitive verifications of such statements as that any string of strokes can be extended only if one carries them out on the basis of specific concepts, such as that of a string of strokes. If that were not so, they would not confer any generality." (p. 174) (4) "Although intuition yields one essential element of the idea that there are, at least potentially, infinitely many strings . . . more is involved in the idea, in particular that the operation of adding an additional stroke can be indefinitely iterated. The sense, if any, in which intuition tells us that is not obvious." (p. 176) But (5) "Once one has seen that every string can be extended, it is still another question whether the string resulting by adding another symbol is a different string from the original one. For this it must be of a different type, and it is not obvious why this must be the case. . . . Although it will follow from considerations advanced in Chapter 7 that it is intuitively known that every string can be extended by one of a different type, ideas connected with induction are needed to see it" (p. 178).

There's a lot to be said about all that, though (4) and (5) already indicate that Parsons thinks that, by itself, 'intuition' of stroke-strings might not take us terribly far. But does it take us even as far as Parsons says? For surely it is *not* the case that imagining/intuiting adding a stroke to an inexplicitly articulated string, together with the exercise of the concept of a string of strokes, suffices to give us the idea that any string can be extended. For we can surely conceive of a particularist reasoner, who has the concept of a string, can bring various arrays (more or less explicitly articulated) under that concept, and given a string can recognize that *this* one can be extended – but who can't advance to frame the thought that they can *all* be extended? The generalizing move surely requires a further *thought*, not given in intuition.

Indeed, we might now wonder quite what the notion of intuition is doing here at all. For note that (1) and (2) are a claims about what is *imaginable*. But if we *can* get to general results about extensibility by *imagining* particular strings (or at any rate, imagining strings "leaving inexplicit their articulation into single strokes", thus perhaps $||\ldots||$ with a blurry filling) and then bring them under concepts and generalizing, why do we also need to think in terms of having cognitive access to something *else* which is intrinsically general, i.e. stroke-string types? It seems that Parsonian intuitions actually drop out of the picture. What gives them an essential role in the story?

Finally, note Parsons's pointer forward to the claim that ideas "connected with induction" can still be involved in what is 'intuitively known". We might well wonder again as we did before: what integrity is left to the notion of intuitive knowledge once it is no longer tightly coupled with the idea of some quasi-perceptual source and allows inference, now even non-logical inference, to preserve intuitive knowledge? I can't wrestle with this issue further here: but Parsons ensuing discussion of these matters left me puzzled and unpersuaded.

I have concentrated on just two familiar Parsonian themes, though there's a lot more to be said even about them: and I should emphasize again that there is a great deal in this rich and rewarding book which I am having to pass over in silence – e.g. there's a lot more about the kinds of thoughts beyond the simplest 'intuitive' arithmetic, there's an interesting discussion of the source of our conviction of the uniqueness of the natural number structure, and another treatment of a further Parsonian theme, the supposed impredicativity of the notion of the natural numbers.[6] There's much that will provoke thought and perhaps disagreement. Certainly, no reader with an interest in the philosophy of mathematics can possibly fail to profit from engaging critically with the characteristically subtle twists and turns of Parsons's discussions.

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[6] See my http://www.logicmatters.net for a very much longer section-by-section discussion, covering the whole book.