Defending the Axioms: On the Philosophical Foundations of Set Theory, by Penelope Maddy. Oxford: Oxford University Press, 2011. Pp. x + 150. H/b £25.

In this short and very engagingly written book, Penelope Maddy returns to her longstanding interest in set theory, and does so from the point of view of the main character of her previous book (*Second Philosophy*, Oxford: Oxford University Press, 2007). This enquirer—the Second Philosopher—wants to discover what the world is like, and uses what would be typically described as 'scientific methods' in her endeavour. Unlike a narrowly specializing scientist, however, the Second Philosopher is also interested—though still in a very broadly scientific way—in wider questions of a more 'philosophical' kind: questions concerning, for instance, the reliability of her methods or the nature of human practices of enquiry.

Now, in the course of her wide-ranging investigations, the Second Philosopher will no doubt employ a great deal of mathematics, whose methods appear to be rather different from those of the empirical sciences. This raises a number of (second-)philosophical questions: What is mathematics? Why is it useful? And what are its proper methods?

Maddy's aim is to answer these questions with regard to set theory. To do so, she says, it is important to keep in mind how set theory emerged. The tale she tells in chapter 1 is a familiar one. During the course of the nineteenth century, mathematics developed in such a way as to become autonomous from its applications. As a result, science could no longer provide guidance as to which mathematical entities exist and which mathematical methods are in order. Eventually, cutting a long story short, the role of final arbiter on questions of proof and existence in mathematics is taken on by set theory. This narrative goes some way towards explaining Maddy's focus on set theory. For she agrees that set theory *can* be thought of as being a foundation for mathematics in the sense of being a final arbiter; accordingly, many philosophical issues of truth and existence in mathematics become for her issues of truth and existence in set theory.

In chapter 2, Maddy briefly addresses the question of what the proper methods of set theory are. She considers a number of examples, ranging from Cantor's original introduction of sets, through the justification of Choice, to Woodin's case for Projective Determinacy. The unifying theme is that these methods are proper because they are effective means to achieve specific mathematical goals. The discussion here is very much reminiscent of her *Naturalism in Mathematics* (Oxford: Oxford University Press, 1997).

But in pursuing our mathematical goals, are we in the business of seeking *truths* about a special subject matter, sets? Well, why not say so? Set theory is a deep and fruitful mathematical practice, highly successful in its own terms, which has emerged naturally out of 'the defining empirical inquiry' from which the second philosopher begins (p. 70). There is no broadly scientific reason to stop pursuing it. So as good second philosophers, selfconsciously and reflectively extending and revising our story of the world as best we can, what's not to like? We should therefore happily agree that there are sets, and that set theory tells us about them—but also we should acknowledge that there is no more to be said about them than what set theory asserts. Or so claims the Thin Realist introduced in chapter 3.

A realist of a more traditional philosophical stamp won't be content with that. He is perhaps gripped by an extra-mathematical picture of the sets as genuinely 'out there' quite independently of the natural world, forming a parallel world of entities sitting in a platonic heaven, with a great gulf fixed between the mathematical abstracta and the sublunary world. The harder he pushes that picture, the tougher it is for the Robust Realist to account for why we should think that the methods we sublunary mathematicians use should be a reliable guide to the lie of the land beyond the great gulf. It can then seem that our methods need backing up by some kind of certification that they will deliver the epistemic goods (and what could *that* look like?). Maddy, by contrast, thinks that the first-philosophical demand for such a certification is misplaced: as a second philosopher, working away in the thick of our best practices in science and maths, she wonders why we should suppose that that perfectly ordinary mathematical reasoning should stand in need of the sort of external supplementation that a Robust Realist seems to require.

It is important for Maddy's Thin Realist, however, that our set theory—however wildly abstract it seems—has its connections to less abstract applicable mathematics. Set theory, the rather Quinean thought goes, is an outlying but still not entirely disconnected part of a network of enquiry with empirical anchors. But we might wonder whether the Thin Realist protests too much. To be sure, looking at the historical emergence of modern mathematics, we can trace (as Maddy does in her opening chapter) the long slow emergence from its roots in mathematized science of a purely abstract study driven increasingly by a purely mathematical curiosity: the narrative does bring out some albeit very stretched lines of connection. But starting from where we now are, the picture changes quite sharply: here are the feet-on-the-ground physicists using one bunch of methods, and over there are the modern set theorists doing their improvisatory thing with a very different bunch of rules of play. We might ask: whatever the historical route by which they reached the present situation, is there really still a sense, however stretched, in which the physicists and the set theorists remain in the same business, so that we can sensibly talk of them both as trying to 'uncover truths'?

The new suggestion, explored in chapter 4, is that mathematicians have such very different aims and methods that it serves no good purpose for the second philosopher to say that the mathematicians too are talking about things that 'exist' (sets), or that set-theoretic claims are 'true'. And note that it isn't that the mathematicians should now be thought of as trying to talk about existents but failing: to repeat, the idea is that they just aren't in the same world-tracking game. No wonder, then, that—as Maddy puts it on behalf of such a difference-emphasizing philosopher—our 'well-developed methods of confirming existence and truth aren't even in play here' (p. 89). She calls this second line, according to which set theory isn't in the truth game, ‘Arealism’. (Note that Maddy rather conservatively resists those who would treat ‘true’ as topic-neutral in the spirit of the deflationist or Wright’s minimalist: for her, truth properly so called requires some residual empirical anchoring to the discourse.)

What’s it to be for her second philosopher, then, Thin Realism or Arealism? What’s to choose? In the end, *nothing* according to Maddy. Here’s modern science and its methods; here’s modern maths and *its* methods; here’s the developmental story making a chain of connections; here are the radical differences between the far end points. Look at it one way, and just enough unity can be seen, and then we’ll incline to be realists (at least in some rather thin sense) across the board. Look at it all another way, and a disunity of subject matter and methodology in modern practice will be foregrounded, and (so the story goes) Arealism becomes more attractive. There’s no right answer. Rather, this is all supposed to show that the very notions of ‘truth’ and ‘existence’ are rather less pivotal than philosophers of mathematics have traditionally supposed.

The danger in downplaying ideas of truth and existence is that mathematics comes to be seen as a game without any objective anchoring at all: but surely there is something more to it than that. But Maddy goes on, particularly in chapter 5, to suggest that it isn’t ontology that underpins the objectivity of mathematics and provides a check on our practice: it is not ‘a remote metaphysics that we access through some rational faculty, but the entirely palpable facts of mathematical depth’ (p. 137). And while ‘[a] mathematician may blanch and stammer, unsure of himself, when confronted with questions of truth and existence, but on judgements of mathematical importance and depth he brims with conviction’ (p. 117). So ‘[t]he objective “something more” our set-theoretic methods track is the underlying contours of mathematical depth’ (p. 82). This, perhaps, is the key novel turn in Maddy’s thought in this book.

What are we to make of all this? Here we have space for just two strands of comment.

(A) It is notable that, not very thinly, Maddy’s Thin Realist still takes the Continuum Hypothesis (CH) to have a determinate truth-value. Her argument couldn’t be more straightforward: ‘“CH or not-CH” is a theorem, established by her best methods as a fact about V [the universe of sets]; therefore, CH is either true or false there.’ (p. 62–63) Her Arealist will also endorse ‘CH or not-CH’ for similar reasons.

Note however that the claim that set theory is about a unique universe of sets is crucial to the argument. For otherwise the Law of Excluded Middle would not guarantee that CH is either true or false: CH could hold in some universes and not in others (just as the Commutativity of Addition holds in certain groups and not in others despite the relevant instance of the Excluded Middle being a theorem of group theory). So why believe, from a second-philosophical point of view, that set theory is in the business of establishing facts about a single universe V? Because of set theory’s goal of providing a foundation for classical mathematics, says Maddy: given this goal, set theory strives to describe, as accurately as possible, the single structure V (p. 80).

However, one might riposte that the goal of providing a foundation for mathematics is served just as well by taking set theory to be about a variety of universes—one for each ZFC model, perhaps. For arguably in each of these universes we can find all the mathematical objects we need: groups, topological spaces, etc. And since the different universes are all models of ZFC (which we think is enough to settle ordinary mathematical questions about groups, topological spaces, etc.), the differences between them won't matter at the level of everyday mathematics. Moreover, one might continue, the depth of mathematical facts—in the guise of the large and diverse array of ZFC models obtained via forcing or canonical inner models—has shown that it is not mathematically fruitful to try to describe one privileged ZFC model. It is much more fruitful to take set theory to be about a variety of universes so as to gain a better understanding of the various models of set theory and their connections. (A position of this kind is developed by Joel Hamkins in his 'The Set-Theoretic Multiverse', *Review of Symbolic Logic*, forthcoming.)

Needless to say, this line of thought is still entirely second-philosophical: it considers our mathematical goals to be shaped by the facts of mathematical depth (see p. 82). What Maddy seems to need to offset it, then, is some reason for thinking that the mathematical depth displayed by, e.g., the flexibility of forcing is trumped by that illustrated by, for instance, the stable consequences of the large cardinal hierarchy: and nothing in the notion of depth itself seems available to do the work. In the meantime, the Robust Realist is likely to stand by her views about V and the concept of set, which, she thinks, are what are really needed to licence the belief that CH has a determinate truth-value.

(B) That first worry should already begin to make us suspicious about how much work can really be done by appealing to 'depth'. Here's a more general worry about the notion.

Or should we say 'notions', plural? For we see a number of strands here. Maddy herself writes: 'A generous variety of expressions is typically used to pick out the phenomenon I'm after here: mathematical depth, mathematical fruitfulness, mathematical effectiveness, mathematical importance, mathematical productivity, and so on.' (p. 81) And it becomes clear that for her seeking depth/fruitfulness/productivity also goes with valuing richness or breadth in the mathematical world that emerges under the mathematicians' probings.

But does it have to be like that? In a not very remote country, Fefermania let's say, most working mathematicians—the topologists, the algebraists, the combinatorialists and the like—carry on in very much the same way as here; it's just that the mathematicians with 'foundational' interests are an austere lot, who are driven to try to make do with as little as they really need (for that too is a very recognizable mathematical goal). Mathematicians there still value making the unexpected connections we call 'deep', they distinguish important mathematical results from mere 'brilliancies', they explore fruitful new concepts, just like us.

But when they turn to questions of ‘foundations’ they find it naturally compelling to seek minimal solutions, and look for just enough to suitably unify the rest of their practice, putting a very high premium on e.g. low-cost predicative regimentations. Overall, their mathematical culture keeps free invention on a rather tighter rein, and the old hands dismiss the baroquely extravagant set theories playfully dreamt up by their graduate students as unserious recreational games. Can't we rather easily imagine that mind-set being the locally default one? And yet their local Second Philosopher, surveying the scene without first-philosophical prejudices, reflecting on the mathematical methods deployed, may surely still see her local mathematical practice as being in intellectual good order by her lights. Why not?

Suppose Maddy and the Fefermanian Second Philosopher get to meet and compare notes. Will the latter be very impressed by the former's efforts to ‘defend the axioms’ and lure her into the wilder reaches of Cantor's paradise? We doubt it, if Maddy in the end has to rely on her appeal to mathematical depth. For her Fefermanian counterpart will riposte that *her* local mathematicians also value *real* depth (and fruitfulness when that is distinguished from profligacy): it is just that they also strongly value cleaving more tightly to what is really *needed* by way of rounding the mainstream mathematics they share with us. Who is to say which practice is ‘right’? The Maddy of *Naturalism in Mathematics*, who defended a methodological injunction to ‘maximize’ (to look for powerful and fruitful axioms), would of course think that the Fefermanians will simply be going wrong if they instead put too high a value of working with modest foundational axioms. Our present point is that it is not at all obvious that valuing *depth* should lead us to maximize.

We agree that there is depth to the phenomenon of mathematical depth: all credit to Maddy for inviting philosophers of mathematics to think hard about its role in mathematical practice. But we very much doubt that such a phenomenon can bear the burden that she now finds herself placing on it: if there is objectivity to be had in settling on our set-theoretic axioms, it will arguably need to be rooted in something less malleable, less contestable.

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