Exercises: Elementary Arithmetics

This set of exercises is on theories of arithmetic which *lack* a principle of induction – i.e. theories like Baby Arithmetic and Robinson Arithmetic. We also look at wffs of different 'quantifier complexity'.

Reading

1. *IGT2*, Chs 10, 11.

Exercises

1. (a) Make the proof of Theorem 10.2 (for any m, n, Baby Arithmetic proves $\overline{\mathsf{m}} + \overline{\mathsf{n}} = \overline{\mathsf{m} + \mathsf{n}}$) more rigorous by recasting it as a proof by induction on n.
   (b) Baby Arithmetic only knows about the successor, addition and multiplication functions. How would you expand this theory to a similar Baby Arithmetic Plus, which also knows about exponentiation?
   (c) Show the resulting theory is also negation-complete. (You don't need to repeat all the steps of the corresponding proof for the original version of Baby Arithmetic in §10.2: just think what needs to be added!)
   (d) How can Baby Arithmetic and Baby Arithmetic Plus both be negation-complete theories when one proves more than the other?

2. Complete the description in §10.5 of a deviant model of Q, to give an interpretation on which all the axioms are true but where the following are all false: $\forall \mathsf{x}(\mathsf{Sx} \neq \mathsf{x})$, $\forall \mathsf{x}(0 + \mathsf{x} = \mathsf{x})$, $\forall \mathsf{x}(0 \times \mathsf{x} = 0)$, and $\forall \mathsf{x}\forall \mathsf{y}\forall \mathsf{z}(\mathsf{x} \times (\mathsf{y} \times \mathsf{z}) = (\mathsf{x} \times \mathsf{y}) \times \mathsf{z})$.

3. In §11.3, we claimed that Q is 'order-adequate'. To prove this involves establishing nine claims, of which four – labelled (O1), (O2), (O3) and (O9) – are indeed proved in §11.8. Refresh your memory of those cases, and then as a warm-up exercise, show that
   (a) For any n, $\mathsf{Q} \vdash \forall \mathsf{x}(\mathsf{Sx} + \overline{\mathsf{n}} = \mathsf{x} + \mathsf{S}\overline{\mathsf{n}})$.

   And then prove the other five (O) claims (preferably by again informally sketching natural deduction arguments). That is to say, show that
   (b) For any n, if $\mathsf{Q} \vdash \varphi(0)$, $\mathsf{Q} \vdash \varphi(1)$, ..., $\mathsf{Q} \vdash \varphi(\overline{\mathsf{n}})$, then $\mathsf{Q} \vdash (\forall \mathsf{x} \leq \overline{\mathsf{n}})\varphi(\mathsf{x})$.
   (c) For any n, if $\mathsf{Q} \vdash \varphi(0)$, or $\mathsf{Q} \vdash \varphi(1)$, ..., or $\mathsf{Q} \vdash \varphi(\overline{\mathsf{n}})$, then $\mathsf{Q} \vdash (\exists \mathsf{x} \leq \overline{\mathsf{n}})\varphi(\mathsf{x})$.
   (d) For any n, $\mathsf{Q} \vdash \forall \mathsf{x}(\mathsf{x} \leq \overline{\mathsf{n}} \to \mathsf{x} \leq \mathsf{S}\overline{\mathsf{n}})$.
   (e) For any n, $\mathsf{Q} \vdash \forall \mathsf{x}(\overline{\mathsf{n}} \leq \mathsf{x} \to (\overline{\mathsf{n}} = \mathsf{x} \lor \mathsf{S}\overline{\mathsf{n}} \leq \mathsf{x}))$.
   (f) For any $n > 0$, $\mathsf{Q} \vdash (\forall \mathsf{x} \leq \overline{\mathsf{n} - 1})\varphi(\mathsf{x}) \to (\forall \mathsf{x} \leq \overline{\mathsf{n}})(\mathsf{x} \neq \overline{\mathsf{n}} \to \varphi(\mathsf{x}))$.

4. Revise the definitions of expressing and capturing properties in §§5.4, 5.5. And as a preliminary task,
   (a) Confirm that $\mathsf{Q} \vdash (\exists \mathsf{x} \leq \overline{\mathsf{n}})\varphi(\mathsf{x}) \leftrightarrow (\varphi(0) \lor \varphi(1) \lor \varphi(2) \lor \ldots \lor \varphi(\overline{\mathsf{n}}))$.

   Then,

(b) Find L_A wffs whose only quantifiers are bounded quantifiers which *express* the properties of
   i. being an even number [use only addition and a bounded quantifier],
   ii. being a square number,
   iii. being a prime number.

(c) Use the wffs in your answers to (a) and (b) to show that Q can *capture* the properties of
   i. being an even number,
   ii. being a square number,
   iii. being a prime number.

5. We now meet another weak, finitely axiomatized arithmetic.

   Suppose Q^* is the new theory whose language is L_A plus '$\leq$' as a built-in two-place relation, whose logic is still first-order logic, and whose axioms are those of Q *except for Axiom 3*, together with these new axioms:

   Axiom 8 $\forall x(x \leq 0 \rightarrow x = 0)$,

   Axiom 9 $\forall x\forall y(x \leq Sy \leftrightarrow (x \leq y \vee x = Sy))$,

   Axiom 10 $\forall x\forall y(x \leq y \vee y \leq x)$.

   (a) Show that Q^* satisfies conditions (O1) to (O6) and (O8) for being order-adequate.
   (b) Show that Q^* is Σ_1-complete.
   (c) Find a theorem which is a theorem of Q^* but not a theorem of Q augmented with the usual definition of '$\leq$'. [Hint: you know from Question 2 one way of showing that something is not a Q-theorem.]
   (d) Find a theorem which is a theorem of Q but which is not a theorem of Q^*. [Hint: this is easy if you know just a little about the theory of ordinals.]

6. Determine whether the following wffs of L_A are Δ_0, Σ_1, Π_1 or none of those.
   (a) $\neg(S0 + SS0) = SS0$,
   (b) $\forall x(x + 0 = x)$,
   (c) $(\forall x \leq SSS0)(x + 0 = x)$,
   (d) $\exists y(\forall x \leq y)(x + y = z)$,
   (e) $(\forall x \leq y)\exists y(x + y = z)$,
   (f) $(\forall x \leq y)\neg\exists y(x + y = z)$,
   (g) $x \leq y \rightarrow \exists z(x + z = y)$,
   (h) $\forall x(x \leq y \rightarrow \forall z(x + y = z))$,
   (i) $\forall y(\exists x \leq SSSSS0)(x + x \leq y)$,
   (j) $\exists x\, x = y \vee \exists z(x + z = y)$,
   (k) $\exists x(x = y \vee \exists z(x + z = y))$,
   (l) $x = SS0 \vee \forall y(y \leq (x + 1) \times 2)$
   (m) $\forall x(\exists y \leq SSSSSS0)\forall z(x \times (y \times z) = x \times z)$
   (n) $\forall x\exists y\forall z(x \times (y \times z) = x \times z)$

   Also, turn the sketched proof of Theorem 11.4 (ii) into a proper proof by course-of-values induction over the degree of complexity of Σ_1 wffs.

7. Bookwork revision! Now close *IGT2*, and outline proofs of the following key results:
   (a) Q is Σ_1 complete.
   (b) A Π_1 sentence is true if and only if that sentence is consistent with Q.
   (c) If a theory extends Q, it is consistent if and only if it is Π_1 sound.