Solutions: Elementary Arithmetics

1. (a) Make the proof of Theorem 10.2 (for any m, n, Baby Arithmetic proves $\overline{\mathsf{m}} + \overline{\mathsf{n}} = \overline{\mathsf{m} + \mathsf{n}}$) more rigorous by recasting it as a proof by induction on n.

   (b) Baby Arithmetic only knows about the successor, addition and multiplication functions. How would you expand this theory to a similar Baby Arithmetic Plus, which also knows about exponentiation?

   (c) Show the resulting theory is also negation-complete. (You don't need to repeat all the steps of the corresponding proof for the original version of Baby Arithmetic in §10.2: just think what needs to be added!)

   (d) How can Baby Arithmetic and Baby Arithmetic Plus both be negation-complete theories when one proves more than the other?

(a) Choose some arbitrary m to keep fixed through the argument, and let $\varphi(n)$ hold when $\mathsf{BA} \vdash \overline{\mathsf{m}} + \overline{\mathsf{n}} = \overline{\mathsf{m} + \mathsf{n}}$.

It is trivial that (i) $\varphi(0)$ – for $\mathsf{BA} \vdash \overline{\mathsf{m}} + \mathsf{0} = \overline{\mathsf{m}}$.

So now suppose $\varphi(n)$. By hypothesis, then, there is a proof that finishes

$$\overbrace{\mathsf{SS}\ldots\mathsf{S}}^{m}\mathsf{0} + \overbrace{\mathsf{SS}\ldots\mathsf{S}}^{n}\mathsf{0} = \overbrace{\mathsf{SS}\ldots\mathsf{S}}^{m+n}\mathsf{0}$$

But now we can continue the proof by invoking an instance of Schema 4:

$$\overbrace{\mathsf{SS}\ldots\mathsf{S}}^{m}\mathsf{0} + \overbrace{\mathsf{SS}\ldots\mathsf{S}}^{n+1}\mathsf{0} = \mathsf{S}(\overbrace{\mathsf{SS}\ldots\mathsf{S}}^{m}\mathsf{0} + \overbrace{\mathsf{SS}\ldots\mathsf{S}}^{n}\mathsf{0})$$

Now use Leibniz's Law. Invoke the first identity to substitute on the right in the second identity, to get

$$\overbrace{\mathsf{SS}\ldots\mathsf{S}}^{m}\mathsf{0} + \overbrace{\mathsf{SS}\ldots\mathsf{S}}^{n+1}\mathsf{0} = \mathsf{S}(\overbrace{\mathsf{SS}\ldots\mathsf{S}}^{m+n}\mathsf{0}) = \overbrace{\mathsf{SS}\ldots\mathsf{S}}^{m+n+1}\mathsf{0}$$

which shows that $\varphi(n+1)$.

Whence (ii) for all n, if $\varphi(n)$, then $\varphi(n+1)$.

By induction, therefore, (i) and (ii) give us (iii): $\varphi(n)$ for any n. That is to say, for our chosen m, we have $\mathsf{BA} \vdash \overline{\mathsf{m}} + \overline{\mathsf{n}} = \overline{\mathsf{m} + \mathsf{n}}$ for any n. But m was arbitrary, so that establishes Theorem 10.2.

(b) Recall, the exponential m^n is such that $m^0 = 1$ and $m^{n+1} = m^n \cdot m$. And those two equations allow us (given we already know about multiplication) to compute m^n for any natural numbers m, n.

If we are going to express this in an extension of Baby Arithmetic, BA, then obviously we need to add some notation. To keep things looking familiar, let's follow informal mathematical practice and allow ourselves to write in our formal theory '$\overline{\mathsf{m}}^{\overline{\mathsf{n}}}$' as a term denoting m^n: call the resulting of augmenting the language of Baby Arithmetic with this extra notation $\mathcal{L}_{B+}$. Then we'll say that the axioms of Baby

Arithmetic Plus, BA^+, are all the old axioms of BA together with any instances of the following further schemata:

Schema 7 $\overline{\mathsf{m}}^{\mathsf{0}} = \mathsf{S0}$
Schema 8 $\overline{\mathsf{m}}^{\mathsf{S\overline{n}}} = \overline{\mathsf{m}}^{\overline{\mathsf{n}}} \times \overline{\mathsf{m}}$.

(c) As a check, show using our schemata that $\mathsf{BA}^+ \vdash \mathsf{SSS0}^{\mathsf{SS0}} = \mathsf{SSSSSSSSS0}$. Then, in the spirit of the arm-waving proofs of §10.1, we can just note that in a similar way we have $\mathsf{BA}^+ \vdash \overline{\mathsf{m}}^{\overline{\mathsf{n}}} = \overline{\mathsf{m}^{\mathsf{n}}}$ for any m, n.

Theorem 10.4 now goes through in the revised form, *If τ is a term of $\mathcal{L}_{B^+}$, the expanded language of* BA^+*, which takes the value t on the intended interpretation of that language, then* $\mathsf{BA}^+ \vdash \tau = \overline{\mathsf{t}}$. You just need to add to the proof a clause (iv) that if BA^+ correctly evaluates σ and τ it will correctly evaluate σ^τ. Then the rest of proof of negation completeness will go through exactly as before.

(d) To say a theory T is negation complete is to say that for every sentence φ of T's language, either $T \vdash \varphi$ or $T \vdash \neg\varphi$ (§4.4). So to say that BA is negation complete is to say that for every sentence of its language $\mathcal{L}_B$ it decides the sentence one way or the other. To say that BA^* is negation complete is to say that for every sentence of its richer language $\mathcal{L}_{B^*}$ it decides the sentence one way or the other. There's no conflict here! Sure, BA^* proves more than BA proves: but that doesn't mean that BA falls short in what it proves in its own language – the new results that BA^* proves all involve exponentiation.

2. Complete the description in §10.5 of a deviant model of Q, to give an interpretation on which all the axioms are true but where the following are all false: $\forall \mathsf{x}(\mathsf{Sx} \neq \mathsf{x})$, $\forall \mathsf{x}(\mathsf{0} + \mathsf{x} = \mathsf{x})$, $\forall \mathsf{x}(\mathsf{0} \times \mathsf{x} = \mathsf{0})$, and $\forall \mathsf{x}\forall \mathsf{y}\forall \mathsf{z}(\mathsf{x} \times (\mathsf{y} \times \mathsf{z}) = (\mathsf{x} \times \mathsf{y}) \times \mathsf{z})$.

Recall our model in §10.5 has as domain $\{0, 1, 2, 3, \ldots, a, b\}$, i.e. the natural numbers plus two new distinct rogue elements.

We interpreted '$\mathsf{0}$' as still denoting 0, and 'S' as expressing the successor* function S^*, where $S^*n = Sn$ for natural numbers in the domain, $S^*a = a$, $S^*b = b$. It is easily checked that Axioms 1 to 3 of Q still hold on this interpretation. But of course $\forall \mathsf{x}(\mathsf{Sx} \neq \mathsf{x})$ fails.

We then interpreted '+' as expressing the addition* function $+^*$, where $+^*$ agrees with $+$ on two natural numbers, $a +^* n = a$, $b +^* n = b$, while for any x in the domain, $x +^* a = b$ and $x +^* b = a$. In tabular form, we have

$+^*$	n	a	b
m	$m+n$	b	a
a	a	b	a
b	b	b	a

(An entry gives the result of taking the item at the beginning of its row and adding* the item at the top of its column.) Axioms 4 and 5 of Q now also still hold – check this claim! But since $0 +^* a = b \neq a$, $\forall \mathsf{x}(\mathsf{0} + \mathsf{x} = \mathsf{x})$ comes out false.

So far that's just repeating what's in §10.5: we now need to add an interpretation of '$\times$'. So we interpret this as expressing some multiplication* function $\times^*$. Since successor* and addition* work normally on normal numbers, we will naturally want multiplication* to work normally there too, i.e. we will require $\times^*$ to agree with $\times$ when applied to two natural numbers.

To conform to Axiom 6, we must have $a \times^* 0 = 0$ and $b \times^* 0 = 0$.

To conform to Axiom 7, $a \times^* Sm = a \times^* m + a$. But whatever $a \times^* m$ is, adding a gives b as we've just seen. So $a \times^* n = b$ for $n > 0$, and likewise $b \times^* n = a$ for $n > 0$.

For clarity's sake, let's again put what we've got so far in tabular form:

$\times^*$	0	$n\ (>0)$	a	b
0	0	0		
$m\ (>0)$	0	$m \times n$		
a	0	b		
b	0	a		

Now, recall that $a = S^*a$, hence $a \times^* a = a \times^* S^*a$. But by Axiom 7, $a \times^* S^*a = (a \times^* a) +^* a$. But adding* a to anything gives b. Hence $a \times^* a = b$. Similarly $a \times^* b = a \times^* S^*b = (a \times^* b) +^* a = b$. Interchanging as and bs gives us dual results, so we are now have:

$\times^*$	0	$n\ (>0)$	a	b
0	0	0		
$m\ (>0)$	0	$m \times n$		
a	0	b	b	b
b	0	a	a	a

Almost there! But how, if at all, do the axioms of Q constrain the value of $0 \times^* a$? We know from what's gone before that $0 \times^* a = 0 \times^* S^*a = (0 \times^* a) +^* 0 = 0 \times^* a \ldots$. However that isn't very helpful! In fact, in this top right corner there are choices to be made. It can be checked, for example, that this definition for $\times^*$ will work fine, and satisfy our axioms:

$\times^*$	0	$n\ (>0)$	a	b
0	0	0	a	b
$m\ (>0)$	0	$m \times n$	a	b
a	0	b	b	b
b	0	a	a	a

And as required, that will give a model on which $\forall \mathsf{x}(0 \times \mathsf{x} = 0)$ is false, since $0 \times^* a \neq 0$. Note too that $(a \times^* b) \times^* a = a$ while $a \times^* (b \times^* a) = b$: so $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}(\mathsf{x} \times (\mathsf{y} \times \mathsf{z}) = (\mathsf{x} \times \mathsf{y}) \times \mathsf{z})$ is false in this model too.

3. In §11.3, we claimed that Q is 'order-adequate'. To prove this involves establishing nine claims, of which four – labelled (O1), (O2), (O3) and (O9) – are indeed proved in §11.8. Refresh your memory of those cases, and then as a warm-up exercise, show that

   (a) For any n, $\mathsf{Q} \vdash \forall \mathsf{x}(\mathsf{Sx} + \overline{\mathsf{n}} = \mathsf{x} + \mathsf{S}\overline{\mathsf{n}})$.

   And then prove the other five (O) claims (preferably by again informally sketching natural deduction arguments). That is to say, show that

   (b) For any n, if $\mathsf{Q} \vdash \varphi(0)$, $\mathsf{Q} \vdash \varphi(1)$, ..., $\mathsf{Q} \vdash \varphi(\overline{\mathsf{n}})$, then $\mathsf{Q} \vdash (\forall \mathsf{x} \leq \overline{\mathsf{n}})\varphi(\mathsf{x})$.

   (c) For any n, if $\mathsf{Q} \vdash \varphi(0)$, or $\mathsf{Q} \vdash \varphi(1)$, ..., or $\mathsf{Q} \vdash \varphi(\overline{\mathsf{n}})$, then $\mathsf{Q} \vdash (\exists \mathsf{x} \leq \overline{\mathsf{n}})\varphi(\mathsf{x})$.

   (d) For any n, $\mathsf{Q} \vdash \forall \mathsf{x}(\mathsf{x} \leq \overline{\mathsf{n}} \rightarrow \mathsf{x} \leq \mathsf{S}\overline{\mathsf{n}})$.

   (e) For any n, $\mathsf{Q} \vdash \forall \mathsf{x}(\overline{\mathsf{n}} \leq \mathsf{x} \rightarrow (\overline{\mathsf{n}} = \mathsf{x} \vee \mathsf{S}\overline{\mathsf{n}} \leq \mathsf{x}))$.

   (f) For any $n > 0$, $\mathsf{Q} \vdash (\forall \mathsf{x} \leq \overline{\mathsf{n-1}})\varphi(\mathsf{x}) \rightarrow (\forall \mathsf{x} \leq \overline{\mathsf{n}})(\mathsf{x} \neq \overline{\mathsf{n}} \rightarrow \varphi(\mathsf{x}))$.

(a) Arguing inside Q, we want to show

$$\mathsf{Sa} + \overbrace{\mathsf{SSS}\ldots\mathsf{S}}^{n\ \mathsf{S}s}\mathsf{0} = \mathsf{a} + \overbrace{\mathsf{SSSS}\ldots\mathsf{S}}^{n+1\ \mathsf{S}s}\mathsf{0}$$

Keep on applying Axiom 5 to prove that $\mathsf{Sa} + \mathsf{SSS}\ldots\mathsf{S0} = \mathsf{S}(\mathsf{Sa} + \mathsf{SS}\ldots\mathsf{S0}) = \mathsf{SS}(\mathsf{Sa} + \mathsf{S}\ldots\mathsf{S0}) = \ldots = \mathsf{SSS}\ldots\mathsf{S}(\mathsf{Sa} + \mathsf{0})$ (moving all the n right-hand occurrences of S one at a time to the left). But the last wff equals $\mathsf{SSS}\ldots\mathsf{SSa}$ by Axiom 4, and hence by Axiom 4 again equals $\mathsf{SSS}\ldots\mathsf{SS}(\mathsf{a} + \mathsf{0})$. Now invoke Axiom 5 again to move all the $n + 1$ occurrences to S to the right, one at a time, to prove that wff in turn equals $\mathsf{a} + \mathsf{SSSS}\ldots\mathsf{S0}$. That establishes the desired equation, and since a was arbitrary, the target generalization follows.

(b) Assume Q proves each of $\varphi(\mathsf{0})$, $\varphi(\mathsf{1})$, $\ldots$, $\varphi(\overline{\mathsf{n}})$. Then, suppose $\mathsf{a} \leq \overline{\mathsf{n}}$, for arbitrary a. By (O3), we have $\mathsf{a} = \mathsf{0} \lor \mathsf{a} = \mathsf{1} \lor \ldots \lor \mathsf{a} = \overline{\mathsf{n}}$. Now we argue by cases: each disjunct separately – combined with one of our initial suppositions – gives $\varphi(\mathsf{a})$; so we can conclude $\varphi(\mathsf{a})$. Discharging our temporary supposition, $\mathsf{a} \leq \overline{\mathsf{n}} \to \varphi(\mathsf{a})$. Generalize, and we are done.

(c) Assume Q proves $\varphi(\overline{\mathsf{k}})$ for some $k \leq n$. Since Q captures *less-than-than-or-equal-to*, Q proves $\overline{\mathsf{k}} \leq \overline{\mathsf{n}} \ \land \ \varphi(\overline{\mathsf{k}})$, hence proves $\exists \mathsf{x}(\mathsf{x} \leq \overline{\mathsf{n}} \ \land \ \varphi(\mathsf{x}))$.

(d) Arguing inside Q, suppose $\mathsf{a} \leq \overline{\mathsf{n}}$ for arbitrary a. Then $\mathsf{a} = \mathsf{0} \lor \mathsf{a} = \mathsf{1} \lor \ldots \lor \mathsf{a} = \overline{\mathsf{n}}$ by (O3). So by trivial logic, $\mathsf{a} = \mathsf{0} \lor \mathsf{a} = \mathsf{1} \lor \ldots \lor \mathsf{a} = \overline{\mathsf{n}} \lor \mathsf{a} = \overline{\mathsf{n+1}}$. So by (O2), we get $\mathsf{a} \leq \overline{\mathsf{n+1}}$, i.e. $\mathsf{a} \leq \mathsf{S}\overline{\mathsf{n}}$. So Q proves $\mathsf{a} \leq \overline{\mathsf{n}} \to \mathsf{a} \leq \mathsf{S}\overline{\mathsf{n}}$ for arbitrary a. Generalize and we are done.

(e) Arguing inside Q, suppose $\overline{\mathsf{n}} \leq \mathsf{a}$ for arbitrary a. Then, by the definition of the order relation, for some b, $\mathsf{b} + \overline{\mathsf{n}} = \mathsf{a}$. By Q's Axiom 3, either (i) $\mathsf{b} = \mathsf{0}$ or (ii) $\mathsf{b} = \mathsf{Sc}$ for some c. If (i), then $\mathsf{0} + \mathsf{SS}\ldots\mathsf{S0} = \mathsf{a}$, and applying Axiom 5 n times and then Axiom 4 gives $\mathsf{SS}\ldots\mathsf{S0} = \mathsf{a}$, i.e. $\overline{\mathsf{n}} = \mathsf{a}$. If (ii), $\mathsf{Sc} + \overline{\mathsf{n}} = \mathsf{a}$. But $\mathsf{Sc} + \overline{\mathsf{n}} = \mathsf{c} + \mathsf{S}\overline{\mathsf{n}}$, by our warm-up result (a) above, hence $\mathsf{c} + \mathsf{S}\overline{\mathsf{n}} = \mathsf{a}$. Hence $\exists \mathsf{v}(\mathsf{v} + \mathsf{S}\overline{\mathsf{n}} = \mathsf{a})$, i.e. $\mathsf{S}\overline{\mathsf{n}} \leq \mathsf{a}$. Therefore either way we get $\overline{\mathsf{n}} = \mathsf{a} \lor \mathsf{S}\overline{\mathsf{n}} \leq \mathsf{a}$, and we are done.

(f) Arguing inside Q, suppose $(\forall \mathsf{x} \leq \overline{\mathsf{n-1}})\varphi(\mathsf{x})$. By (O3), we have, $\varphi(\mathsf{0}) \land \varphi(\mathsf{1}) \land \ldots \ldots \land \varphi(\overline{\mathsf{n-1}})$. So we can trivially prove each of $(\mathsf{0} \neq \overline{\mathsf{n}} \to \varphi(\mathsf{0}))$, $(\mathsf{1} \neq \overline{\mathsf{n}} \to \varphi(\mathsf{1}))$, $\ldots$, $(\overline{\mathsf{n-1}} \neq \overline{\mathsf{n}} \to \varphi(\overline{\mathsf{n-1}}))$. But $\overline{\mathsf{n}} = \overline{\mathsf{n}}$ is also trivial, and hence $(\overline{\mathsf{n}} \neq \overline{\mathsf{n}} \to \varphi(\overline{\mathsf{n}}))$. By (b) above, putting those together entails $(\forall \mathsf{x} \leq \overline{\mathsf{n}})(\mathsf{x} \neq \overline{\mathsf{n}} \to \varphi(\mathsf{x}))$.

4. (a) Confirm that $\mathsf{Q} \vdash (\exists \mathsf{x} \leq \overline{\mathsf{n}})\varphi(\mathsf{x}) \leftrightarrow (\varphi(\mathsf{0}) \lor \varphi(\mathsf{1}) \lor \varphi(\mathsf{2}) \lor \ldots \lor \varphi(\overline{\mathsf{n}}))$.
   (b) Find L_A wffs whose only quantifiers are bounded quantifiers which *express* the properties of
      i. being an even number [use only addition and a bounded quantifier],
      ii. being a square number,
      iii. being a prime number.
   (c) Use the wffs in your answers to (a) and (b) to show that Q can *capture* the properties of
      i. being an even number,
      ii. being a square number,
      iii. being a prime number.

(a) We argue in Q. Suppose $(\exists \mathsf{x} \leq \overline{\mathsf{n}})\varphi(\mathsf{x})$, i.e. unpacking the abbreviation, let's suppose $\exists \mathsf{x}(\mathsf{x} \leq \overline{\mathsf{n}} \land \varphi(\mathsf{x}))$. So for some a, $\mathsf{a} \leq \overline{\mathsf{n}} \land \varphi(\mathsf{a})$. By (O3) in §11.3, we can deduce $(\mathsf{a} = 0 \lor \mathsf{a} = 1 \lor \mathsf{a} = 2 \lor \ldots \lor \mathsf{a} = \overline{\mathsf{n}}) \land \varphi(\mathsf{a})$.

But that in turn implies $\varphi(0) \lor \varphi(1) \lor \varphi(2) \lor \ldots \lor \varphi(\overline{\mathsf{n}})$. So (by Existential Quantifier Elimination and Conditional Proof) that gives us one direction of the biconditional.

For the other direction, assume the right-hand side of the biconditional. We just note that each disjunct $\varphi(\mathsf{m})$ (for $m \leq n$), together with the provable $\overline{\mathsf{m}} \leq \overline{\mathsf{n}}$ gives us $\exists \mathsf{x}(\mathsf{x} \leq \overline{\mathsf{n}} \land \varphi(\mathsf{x}))$. [Reality check: *Why* is $\overline{\mathsf{m}} \leq \overline{\mathsf{n}}$ provable in Q when $m \leq n$?] So an argument by cases followed by another application of Conditional Proof gives us the other direction of the biconditional.

(b) i. A number n is even if it is the sum of m and m, where of course $m \leq n$. So the wff $(\exists \mathsf{y} \leq \mathsf{x})\ \mathsf{x} = \mathsf{y} + \mathsf{y}$ expresses the property of being even.

ii. Similarly a number n is a square if it is the product of m and m, where of course again $m \leq n$. So the wff $(\exists \mathsf{y} \leq \mathsf{x})\ \mathsf{x} = \mathsf{y} \times \mathsf{y}$ expresses the property of being a square.

iii. We gave a formula for expressing the property of being prime in §5.4, which says that x is not 1, and for two factors which multiply to give x, one of them must be 1. But a number's factors must be less than or equal to it. So – as in fact noted in §11.5 – we can express the property of being prime like this:

$$\mathsf{x} \neq 1 \land (\forall \mathsf{u} \leq \mathsf{x})(\forall \mathsf{v} \leq \mathsf{x})(\mathsf{u} \times \mathsf{v} = \mathsf{x} \to (\mathsf{u} = 1 \lor \mathsf{v} = 1)).$$

For future use, abbreviate this wff $\mathsf{P}(\mathsf{x})$.

(c) i. We will show that the same wff not only expresses the property of being even but captures it. So we need to prove

1. If n is even, then $\mathsf{Q} \vdash (\exists \mathsf{y} \leq \overline{\mathsf{n}})\ \overline{\mathsf{n}} = \mathsf{y} + \mathsf{y}$,
2. If n isn't even, then $\mathsf{Q} \vdash \neg(\exists \mathsf{y} \leq \overline{\mathsf{n}})\ \overline{\mathsf{n}} = \mathsf{y} + \mathsf{y}$.

For (1), we just note that if n is even, then for some $m \leq n$, $n = m + m$. Q can then prove $\overline{\mathsf{n}} = \overline{\mathsf{m}} + \overline{\mathsf{m}}$; and Q can also prove $\overline{\mathsf{m}} \leq \overline{\mathsf{n}}$. So putting those together and existentially quantifying, Q proves $\exists \mathsf{y}(\mathsf{y} \leq \overline{\mathsf{n}}) \land \overline{\mathsf{n}} = \mathsf{y} + \mathsf{y})$ as we wanted.

For (2), suppose n isn't even, so for some $m \leq n$, $n = (m + m) + 1$, and therefore Q proves $\overline{\mathsf{n}} = (\overline{\mathsf{m}} + \overline{\mathsf{m}}) + 1$.

Now suppose for reductio that we have $(\exists \mathsf{y} \leq \overline{\mathsf{n}})\ \overline{\mathsf{n}} = \mathsf{y} + \mathsf{y}$. Arguing in Q, by (a), we have

$$\overline{\mathsf{n}} = 0 + 0\ \lor\ \overline{\mathsf{n}} = 1 + 1\ \lor\ \overline{\mathsf{n}} = 2 + 2\ \lor\ \ldots \lor\ \overline{\mathsf{n}} = \overline{\mathsf{n}} + \overline{\mathsf{n}}.$$

We then just have to show that each disjunct is inconsistent with $\overline{\mathsf{n}} = (\overline{\mathsf{m}} + \overline{\mathsf{m}}) + 1$. The idea is that if $\overline{\mathsf{n}} = \overline{\mathsf{k}} + \overline{\mathsf{k}}$, then $\overline{\mathsf{k}} + \overline{\mathsf{k}} = (\overline{\mathsf{m}} + \overline{\mathsf{m}}) + 1$, and we know that Q can refute any such false equation between particular numbers.

ii. Proved very similarly.

iii. This time we prove the wff we gave as expressing the property of being prime also captures that property. So we need to show

1. If n is prime, then $\mathsf{Q} \vdash \mathsf{P}(\overline{\mathsf{n}})$,
2. If n isn't prime, then $\mathsf{Q} \vdash \neg\mathsf{P}(\overline{\mathsf{n}})$.

with P as in (b)iii above.

For (1), we first note that if n is prime then $n \neq 1$, and so $\mathsf{Q} \vdash \overline{\mathsf{n} \neq 1}$. And also, for every $j, k \leq n$, we have $jk = n \to j = 1 \lor k = 1$. So in each case, $\mathsf{Q} \vdash (\bar{\mathsf{j}} \times \bar{\mathsf{k}} = \bar{\mathsf{n}} \to \bar{\mathsf{j}} = 1 \lor \bar{\mathsf{k}} = 1)$. We then use (O3) from §11.3 to show that $\mathsf{Q} \vdash (\forall \mathsf{v} \leq \bar{\mathsf{n}})(\bar{\mathsf{j}} \times \mathsf{v} = \bar{\mathsf{n}} \to \bar{\mathsf{j}} = 1 \lor \mathsf{v} = 1)$, and then use (O3) again to show $\mathsf{Q} \vdash (\forall \mathsf{u} \leq \bar{\mathsf{n}})(\forall \mathsf{v} \leq \bar{\mathsf{n}})(\mathsf{u} \times \mathsf{v} = \bar{\mathsf{n}} \to (\mathsf{u} = 1 \lor \mathsf{v} = 1)))$. Putting the proofs together gives us $\mathsf{Q} \vdash \mathsf{P}(\bar{\mathsf{n}})$.

For (2), we note that if n isn't prime, then either (a) $n = 1$, or (b) $n = jk$ where $j \neq 1$ and $k \neq 1$.

In case (a) $\mathsf{Q} \vdash \overline{\mathsf{n} = 1}$, and hence $\mathsf{Q} \vdash \neg\mathsf{P}(\bar{\mathsf{n}})$.

In case (b) Q proves each of $\bar{\mathsf{n}} = \bar{\mathsf{j}} \times \bar{\mathsf{k}}$ and $\bar{\mathsf{j}} \neq 1$ and $\bar{\mathsf{k}} \neq 1$. So by propositional logic we can derive $\neg(\bar{\mathsf{n}} = \bar{\mathsf{j}} \times \bar{\mathsf{k}} \to (\bar{\mathsf{j}} = 1 \lor \bar{\mathsf{k}} = 1)$. Existentially quantify twice to show that $\mathsf{Q} \vdash (\exists \mathsf{u} \leq \bar{\mathsf{n}})(\exists \mathsf{v} \leq \bar{\mathsf{n}})(\mathsf{u} \times \mathsf{v} = \bar{\mathsf{n}} \to (\mathsf{u} = 1 \lor \mathsf{v} = 1))$. Which easily shows $\mathsf{Q} \vdash \neg\mathsf{P}(\bar{\mathsf{n}})$.

So, either way, as we want, $\mathsf{Q} \vdash \neg\mathsf{P}(\bar{\mathsf{n}})$.

5. We now meet another weak, finitely axiomatized arithmetic.

Suppose Q^* is the new theory whose language is L_A plus '$\leq$' as a built-in two-place relation, whose logic is still first-order logic, and whose axioms are those of Q *except for Axiom 3*, together with these new axioms:

Axiom 8 $\quad \forall \mathsf{x}(\mathsf{x} \leq 0 \to \mathsf{x} = 0)$,

Axiom 9 $\quad \forall \mathsf{x}\forall \mathsf{y}(\mathsf{x} \leq \mathsf{Sy} \leftrightarrow (\mathsf{x} \leq \mathsf{y} \lor \mathsf{x} = \mathsf{Sy}))$,

Axiom 10 $\quad \forall \mathsf{x}\forall \mathsf{y}(\mathsf{x} \leq \mathsf{y} \lor \mathsf{y} \leq \mathsf{x})$.

(a) Show that Q^* satisfies conditions (O1) to (O6) and (O8) for being order-adequate.

(b) Show that Q^* is Σ_1-complete.

(c) Find a theorem which is a theorem of Q^* but not a theorem of Q augmented with the usual definition of '$\leq$'. [Hint: you know from Question 2 one way of showing that something is not a Q-theorem.]

(d) Find a theorem which is a theorem of Q but which is not a theorem of Q^*. [Hint: this is easy if you know just a little about the theory of ordinals.]

(a) (O1) Arguing inside Q^*: Ax. 10 gives $\mathsf{a} \leq 0 \lor 0 \leq \mathsf{a}$ for arbitrary a. By Ax. 8, the first disjunct gives $\mathsf{a} = 0$, and since A. 10 also gives $0 \leq 0$, we then have $0 \leq \mathsf{a}$. So *both* disjuncts yield $0 \leq \mathsf{a}$, and since a was arbitrary, we can generalize.

(O2) Q^* proves $\mathsf{a} = 0 \to 0 \leq \mathsf{a}$ for arbitrary a, and hence proves $\forall \mathsf{x}(\mathsf{x} = 0 \to \mathsf{x} \leq 0)$.

Suppose then that Q^* proves $\forall \mathsf{x}(\{\mathsf{x} = 0 \lor \mathsf{x} = 1 \lor \ldots \lor \mathsf{x} = \bar{\mathsf{n}}\} \to \mathsf{x} \leq \bar{\mathsf{n}})$. Assuming $\mathsf{a} = 0 \lor \mathsf{a} = 1 \lor \ldots \lor \mathsf{a} = \bar{\mathsf{n}} \lor \mathsf{a} = \mathsf{S}\bar{\mathsf{n}}$, we infer $\mathsf{a} \leq \bar{\mathsf{n}} \lor \mathsf{a} = \mathsf{S}\bar{\mathsf{n}}$, and hence by Ax. 9, $\mathsf{a} \leq \mathsf{S}\bar{\mathsf{n}}$. But trivially, $\mathsf{S}\bar{\mathsf{n}} = \overline{\mathsf{n}+1}$. Use conditional proof, generalize on a, and we've shown Q^* proves $\forall \mathsf{x}(\{\mathsf{x} = 0 \lor \mathsf{x} = 1 \lor \ldots \lor \mathsf{x} = \overline{\mathsf{n}+1}\} \to \mathsf{x} \leq \overline{\mathsf{n}+1})$.

In sum, we've shown that the desired result holds for $n = 0$, and that if holds for n it holds for $n + 1$. Induction now gives the desired result.

(O3) Again we proceed by induction. The base case for $n = 0$ is given as Ax. 8. So now assume Q^* proves $\forall \mathsf{x}(\mathsf{x} \leq \bar{\mathsf{n}} \to \{\mathsf{x} = 0 \lor \mathsf{x} = 1 \lor \ldots \lor \mathsf{x} = \bar{\mathsf{n}}\})$. Then suppose $\mathsf{a} \leq \overline{\mathsf{n}+1}$, so by Ax. 9 we have $\mathsf{a} \leq \bar{\mathsf{n}} \lor \mathsf{a} = \overline{\mathsf{n}+1}$, whence by our assumption $\mathsf{a} = 0 \lor \mathsf{a} = 1 \lor \ldots \lor \mathsf{a} = \overline{\mathsf{n}+1}$. Again, conditional proof and generalizing shows Q^* proves $\forall \mathsf{x}(\mathsf{x} \leq \overline{\mathsf{n}+1} \to \{\mathsf{x} = 0 \lor \mathsf{x} = 1 \lor \ldots \lor \mathsf{x} = \overline{\mathsf{n}+1}\})$.

Again, we've shown that the desired result holds for $n = 0$, and that if holds for n it holds for $n + 1$. ...

(O4) Use induction again! For the base case, note that if Q^* proves $\varphi(0)$, then, for arbitrary a it proves $\mathsf{a} \leq 0 \to \varphi(\mathsf{a})$ (since $\mathsf{a} \leq 0 \to \mathsf{a} = 0$). Hence, generalizing on a, if Q^* proves $\varphi(0)$, it proves $(\forall \mathsf{x} \leq 0)\varphi(\mathsf{x})$.

Suppose that if Q^* proves each of $\varphi(0), \varphi(1), \ldots, \varphi(\bar{\mathsf{n}})$, it proves $(\forall \mathsf{x} \leq \bar{\mathsf{n}})\varphi(\mathsf{x})$. Then obviously, if Q^* proves each of $\varphi(0), \varphi(1), \ldots, \varphi(\overline{\mathsf{n}+1})$, it will also prove $\forall \mathsf{x}(\mathsf{x} \leq \bar{\mathsf{n}} \to \varphi(\mathsf{x})) \wedge \varphi(\mathsf{S}\bar{\mathsf{n}}))$, i.e. $\forall \mathsf{x}(\mathsf{x} \leq \bar{\mathsf{n}} \to \varphi(\mathsf{x})) \wedge (\mathsf{x} = \mathsf{S}\bar{\mathsf{n}} \to \varphi(\mathsf{S}\bar{\mathsf{n}})))$, whence $\forall \mathsf{x}((\mathsf{x} \leq \bar{\mathsf{n}} \vee \mathsf{x} \leq \mathsf{S}(\bar{\mathsf{n}})) \to \varphi(\mathsf{S}\bar{\mathsf{n}}))$, hence (by Ax. 9) $\forall \mathsf{x}(\mathsf{x} \leq \mathsf{S}\bar{\mathsf{n}} \to \varphi(\mathsf{S}\bar{\mathsf{n}}))$.

So once again, we've shown that the desired result holds for $n = 0$, and that if holds for n it holds for $n + 1$. Induction now gives the desired result.

(O5) Proved similarly.

(O6) Immediate from Ax. 9.

(O8) Is Ax. 10.

(Note, that in *IGT2* (O7) only features in a proof of (O8), and (O9) doesn't get used later at all: in retrospect, it would have been neater to have dropped (O7) and (O9) as explicit conditions on adequacy.)

(b) Reviewing the proof that Q is Σ_1 complete, it is readily checked that nothing is appealed to which isn't also provable in Q^*.

(c) Note that in our deviant model explored in Question 2, $a = S^*a$; but there is no object o in the model such that $o +^* a = a$, and hence no object o such that $o +^* a = S^*a$. So in the model, (a, a) is an ordered pair of objects falsifies the condition $\mathsf{x} = \mathsf{S}\mathsf{y} \to \mathsf{x} \leq \mathsf{y}$, and so a fortiori falsifies $(\mathsf{x} \leq \mathsf{y} \vee \mathsf{x} = \mathsf{S}\mathsf{y}) \to \mathsf{x} \leq \mathsf{y}$. So Q^*'s Ax. 9 is false in the model, so not a theorem of Q.

(d) For those who know a smidgin about the arithmetic of the ordinals, note that the ordinals satisfy the axioms of Q^* (interpreting successor, addition, multiplication and order in the obvious ways). But this model doesn't satisfy Ax. 3 of Q: a limit ordinal like ω is non-zero but not a successor. Which shows that the axioms of Q^* can't prove Ax. 3 of Q.

Therefore, in so far as we are looking for a finitely axiomatized theory of arithmetic which is Σ_1 complete, we can go for either Q or Q^*, distinct theories each of which is stronger in some respects and weaker in some respects than the other. For an example of a book which uses a version of Q^* by preference, see Boolos, Burgess, and Jeffrey, *Computability and Logic* (4th/5th editions).

6. Determine whether the following wffs of L_A are Δ_0, Σ_1, Π_1 or none of those.

(a) $\neg(\mathsf{S0} + \mathsf{SS0}) = \mathsf{SS0}$,

(b) $\forall \mathsf{x}(\mathsf{x} + 0 = \mathsf{x})$,

(c) $(\forall \mathsf{x} \leq \mathsf{SSS0})(\mathsf{x} + 0 = \mathsf{x})$,

(d) $\exists \mathsf{y}(\forall \mathsf{x} \leq \mathsf{y})(\mathsf{x} + \mathsf{y} = \mathsf{z})$,

(e) $(\forall \mathsf{x} \leq \mathsf{y})\exists \mathsf{y}(\mathsf{x} + \mathsf{y} = \mathsf{z})$,

(f) $(\forall \mathsf{x} \leq \mathsf{y})\neg\exists \mathsf{y}(\mathsf{x} + \mathsf{y} = \mathsf{z})$,

(g) $\mathsf{x} \leq \mathsf{y} \to \exists \mathsf{z}(\mathsf{x} + \mathsf{z} = \mathsf{y})$,

(h) $\forall \mathsf{x}(\mathsf{x} \leq \mathsf{y} \to \forall \mathsf{z}(\mathsf{x} + \mathsf{y} = \mathsf{z}))$,

(i) $\forall y(\exists x \leq SSSSS0)(x + x \leq y)$,
(j) $\exists x\ x = y \lor \exists z(x + z = y)$,
(k) $\exists x(x = y \lor \exists z(x + z = y))$,
(l) $x = SS0 \lor \forall y(y \leq (x + 1) \times 2)$
(m) $\forall x(\exists y \leq SSSSSS0)\forall z(x \times (y \times z) = x \times z)$
(n) $\forall x \exists y \forall z(x \times (y \times z) = x \times z)$

(a) Δ_0 and hence both Σ_1, Π_1 [don't forget that the class of Σ_1 wffs is an expansion of the Δ_0 wffs, and likewise for the Π_1 wffs].

(b) Π_1.

(c) Δ_0 and hence both Σ_1, Π_1.

(d) Σ_1.

(e) Σ_1 [note that the bounded quantification of something Σ_1 is still Σ_1].

(f) None.

(g) None.

(h) Using our definition for bounded quantifiers, that is $(\forall \leq y)\forall z(x + y = z))$, which shows it to be Π_1.

(i) Π_1.

(j) Σ_1.

(k) Σ_1.

(l) Π_1.

(m) Π_1.

(n) None [in fact it is Π_3 – cf. *IGT2*, end of §11.5].

Also, turn the sketched proof of Theorem 11.4 (ii) into a proper proof by course-of-values induction over the degree of complexity of Σ_1 wffs.

Measure the degree of complexity of a wff by the number connectives or the number of (bounded) quantifiers. The result is trivial for the case of wffs of zero complexity. So suppose that the negation of a Σ_1 wff is equivalent to a Π_1 wff, at least up to the case of wffs of complexity n.

Now suppose φ is a Σ_1 wff of complexity $n + 1$. Then φ is either (i) the conjunction or disjunction of two Σ_1 wffs ψ and χ of complexity no more than n, or (ii) it is the bounded quantification of a Σ_1 wff ψ of complexity n, or (iii) it is the unbounded existential quantification of a Σ_1 wff ψ of complexity n.

In case (i) suppose φ is $\psi \land \chi$; then $\neg\varphi$ is equivalent to $\neg\psi \lor \neg\chi$; but by the induction hypothesis, both disjuncts are equivalent to Π_1 wffs, and the disjunction of two Π_1 wffs is still Π_1; so $\neg\varphi$ is also equivalent to a Π_1 wff. Similarly, of course, for the case where φ is $\psi \lor \chi$.

In case (ii) suppose φ is $(\forall\nu \leq \tau)\psi$. Then $\neg\varphi$ is equivalent to $(\exists\nu \leq \tau)\neg\psi$, and by hypothesis $\neg\psi$ is equivalent to a Π_1 wff, and hence so is $\neg\varphi$. Similarly for the other case of bounded quantification.

In case (iii) suppose φ is $\exists\nu\psi$. Then $\neg\varphi$ is equivalent to $\forall\nu\neg\psi$, and by hypothesis $\neg\psi$ is equivalent to a Π_1 wff, and hence so is $\neg\varphi$.

Which shows that if the negation of a Σ_1 wff is equivalent to a Π_1 wff, at least up to the case of wffs of complexity n, then this equivalence also obtains for wffs complexity $n+1$. Course of values induction then allows us to conclude that the equivalence always holds.