# Exercises: Induction

The first six questions concern (various versions of) ordinary arithmetical induction informally introduced in Chapters 9 of *IGT2*.

The remaining four questions broaden the scope by considering other species of induction which you ought to know about as part of your general logical education. Don't worry, however, if you find the later questions a bit taxing – if you like, just go straight to the solutions, treating them as a tutorial.

### Reading

1. *IGT2*, Chs 9.
2. (Optional but useful) Daniel Velleman, *How to Prove It* (CUP, 1994), Ch. 6

### Exercises

1. Just to test basic understanding:
   (a) Show it doesn't matter where you 'start the induction'; i.e. use induction to show that if, for some $k$, $\varphi(k)$ is true and also $(\forall n \geq k)(\varphi(n) \to \varphi(n+1))$, then $(\forall n \geq k)\varphi(n)$.
   (b) *IGT2* states the principle of course-of-values induction as follows: given (i) $\varphi(0)$ and (ii) $\forall n\{(\forall k \leq n)\varphi(k) \to \varphi(n+1)\}$ we can infer (iii) $\forall n\varphi(n)$. Give another version of the principle where the induction starts by making a *single* assumption instead of using both (i) and (ii).
   (c) Without looking back at the reading, show that the simple principle of arithmetical induction and the principle of course-of-values induction imply each other other.

2. This question and the next one are really just *very* elementary mathematics, and I suppose that the solutions are of little conceptual interest. So just answer part (a) of this question by setting out carefully one argument by induction, and then do as many additional parts of these questions as amuse you. (Though if you are a philosopher doing a logic/philosophy of maths course, it will do no harm to scrape some of the rust off your high school maths.)

   Show that, for any (natural number) $n$,
   (a) $1 + 3 + 5 + \ldots + (2n + 1) = (n + 1)^2$
   (b) $1^3 + 2^3 + 3^3 + \ldots + n^3 = [n(n + 1)/2]^2$
   (c) $n^3 - n$ is divisible by 6.

   Also
   (d) By guesswork or otherwise, find a formula for $3^0 + 3^1 + 3^2 + \ldots + 3^n$ and use induction to confirm it is correct.

3. The Fibonacci numbers $F_k$ are defined by: $F_0 = 0$, $F_1 = 1$, and if $k > 1$, $F_k = F_{k-1} + F_{k-2}$. Write down the first few Fibonacci numbers, and then show that, for any $n$,
   (a) $F_0 + F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$

(b) $F_0^2 + F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$

(c) $F_{n-1}F_{n+1} = F_n^2 + (-1)^n$

(It is well worth checking out http://en.wikipedia.org/wiki/Fibonacci_number to find out more about this fascinating sequence.)

4. (The Towers of Hanoi: see picture at http://bit.ly/1c4QSUf.) Suppose you have three posts and a stack of $n$ different sized disks, initially placed on one post with the largest disk on the bottom and with each disk above it smaller than the disk below. You are to move the disks so they end up all on another post, again in decreasing order of size with the largest disk on the bottom. The only moves you are allowed involve taking the top disk from one post and moving it so that it becomes the top disk on another post, without being put on a smaller disk.

   (a) Show that for any $n$ there must be a sequence of moves that does indeed end with all the disks on a post different from the original one in the desired configuration.

   (b) How many moves are required given an initial stack of $n$ disks in the sequence of moves revealed by your answer to the previous question?

5. Use some form of arithmetical induction to show that

   (a) Every wff of your favourite system for the propositional calculus is balanced, i.e. has the same number of left and right parentheses. (We'll assume that you aren't using a Polish bracket-free notation!)

   (b) No proper initial part of a wff (i.e. initial part shorter than the whole wff itself) is itself a wff. [Hint: Assume the desired result holds for wffs with up to $n$ connectives, and then consider whether there could be counterexamples when dealing with a wff with $n + 1$ connectives.]

   Conclude that

   (c) Any non-atomic wff can be decomposed into a main connective and one or two subformulae in exactly one way.

   And then

   (d) Explain how arithmetical induction can be used to give a proof that your favourite deductive system for the classical propositional calculus is sound in the sense that any theorem is a tautology.

6. The Least Number Principle says, informally, that if some number has a given numerical property, then there is a least number with that property.

   (a) Show that the simple induction principle implies and is implied by the Least Number Principle.

   (b) How does the Least Number Principle relate to the principle that any non-empty set of natural numbers has a smallest element?

7. Say that $m$ is fess than $n$ if and only if either (i) $m$ is even and $n$ is odd or (ii) $m$ and $n$ have the same parity (both are even or both are odd).

   (a) How far along the sequence of numbers ordered by fess-than does the number 5 come?

   (b) Formulate and prove a Feast Number Principle.

   (c) How would we formulate a corresponding induction principle? Explain informally why the principle is sound. [Hint: Think of course-of-values induction.]

8. Start with two new definitions:

    1. Let's say that the relation $R$ defined over the objects $X$ is *well-founded* just in case for any non-empty collection of objects $D \subseteq X$, then there is at least one $R$-minimal object $a$ among $D$, i.e. an object $a$ such that $\forall x(x \in D \rightarrow \neg Rxa)$.
    2. Let's say that you can do *w-induction* on the relation $R$ defined over the objects $X$ just in case you can infer $\forall z\varphi(z)$ from $\forall x(\forall y(Ryx \rightarrow \varphi(y)) \rightarrow \varphi(x))$, where the quantifications again are over the objects $X$. [That is, you can infer everything is $\varphi$ from the premiss that if $x$'s $R$-predecessors are all $\varphi$ then so is $x$.]

    Five problems:

    (a) Give non-trivial examples of well-founded relations over three different kinds of objects.

    (b) How does w-induction compare with course-of-values induction over the natural numbers $\mathbb{N}$?

    (c) How does the claim that $<$ is well-founded over $\mathbb{N}$ compare with the Least Number Principle?

    (d) Show that if the relation $R$ defined over $X$ is well-founded then we can do w-induction on the relation $R$. [Hint: prove the contrapositive.]

    (e) Show that if we can do w-induction on the relation $R$ defined over the objects $X$, then $R$ must be well-founded. [Hint: obviously you need to use induction – do the induction on the property an object $x$ has if every $D \subseteq X$ which contains $x$ has an $R$-mininal element.]

    Because we have (d) and (e), what we have temporarily called w-induction is usually called *well-founded induction*

9. Two more definitions.

    1. An *inductive datatype* (a.k.a. a recursive datatype) is defined in terms of some objects $G$, the *generators*, and some functions or operators $O$, the *constructors*, which map an object or objects as input to an object as output. The inductive datatype is then the smallest set (by set-inclusion of course) containing $G$ and closed under the constructors – i.e. if it contains some given objects, then it also contains the results of applying a constructor in $O$ to those objects.
    2. We can do *s-induction* over a set $D$ with respect to generators $G$ (where $G \subseteq D$) and constructors $O$ if and only if, from the premisses (i) that any object in $G$ has property $P$ and (ii) that the constructors preserve property $P$ (i.e applied to things with $P$, a constructor yields something which also has $P$), we can infer (iii) that everything in $D$ has $P$.

    Five problems:

    (a) Show how (1) the natural numbers, (2) the wffs of the propositional calculus, (3) proofs in an axiomatic system of propositional logic, can be regarded as inductive datatypes.

    (b) Show that if $D$ is an inductive datatype, then we can indeed do s-induction over $D$ with respect to its generators and constructors.

    (c) Show conversely, show that if can do s-induction over $D$ with respect to certain generators and constructors, then $D$ is an inductive datatype.

    (d) Use s-induction rather than arithmetical induction to show that every wff of your favourite system of propositional calculus is balanced, i.e. has the same number of left and right parentheses.

(e) Outline a proof using s-induction rather than arithmetical induction to show that every theorem of your favourite system of propositional calculus is a tautology.

Note, what we have temporarily called s-induction is usually called *structural induction*.

10. Given a relation $R$, suppose that $R^*ab$ holds just when either $Rab$ or $\exists x_1(Rax_1 \wedge Rx_1b)$ or $\exists x_1\exists x_2(Rax_1 \wedge Rx_1x_2 \wedge Rx_2b)$ or $\exists x_1\exists x_2\exists x_3(Rax_1 \wedge Rx_1x_2 \wedge Rx_2x_3 \wedge Rx_3b)$ or .... In other words, $R^*ab$ if there is a (finite) chain of $R$-related objects linking $a$ to $b$.

    Suppose, for example, that $Rab$ says that *$a$ is a parent of $b$*. Then the corresponding $R^*ab$ holds when *$a$ is an ancestor of $b$*. For this reason, $R^*$ is said to be the *ancestral* of $R$.

    The *$R$-posterity* of $a$ is the set of objects $x$ such that $R^*ax$.

    A property $P$ is *$R$-hereditary* if, an object $x$ has $P$ and $Rxy$, then $y$ has $P$.

    The *Fregean ancestral* of $R$ is the relation $R^+$ defined as follows: $R^+ab$ if and only if $b$ has every $R$-hereditary property had by every $x$ such that $Rax$.

    (a) Express the definition of the Fregean ancestral of $R$ more formally.

    (b) Show that $R^*ab$ if and only if $R^+ab$.

    (c) Reality check: why isn't the Fregean definition more simply that $R^+ab$ if and only if $b$ has every $R$-hereditary property had by $a$?

    (d) Frege and Russell/Whitehead in effect define the natural numbers as 0 plus its $S$-posterity (meaning posterity with respect to the successor relation $S$ such that $Sxy$ iff $y$ is the immediate successor of $x$). Show it follows from this definition that arithmetical induction is a sound principle.

    (e) How might we generalize this idea of induction over the posterity of a relation? Compare with our definitions of w-induction and s-induction.