31 Informal quantifier rules

This chapter describes four intuitively correct principles for arguing informally with the quantifiers 'every' and 'some' (when they are regimented as quantifier prefixes). Then the next chapter will explain the corresponding rules of inference for arguing with formal QL quantifiers.

31.1 Arguing with universal quantifiers

(a) Let's begin with that hackneyed old favourite,

A Socrates is a man. All men are mortal. Hence Socrates is mortal.

This hardly needs a proof to warrant it! But consider the following derivation, where we render the second premiss using a quantifier-variable notation, with the quantifier prefix running over a suitably inclusive domain:

(1)	Socrates is a man.	(premiss)
(2)	(Everything x is such that) if x is a man, then x is mortal.	(premiss)
(3)	If Socrates is a man, then Socrates is mortal.	(from 2)
(4)	Socrates is mortal.	(from 1, 3)

The principle that is invoked to get line (3) is very straightforward: what applies to everything/everyone in the relevant domain applies to any individual thing in the domain. Using the notion of an 'instance' (introduced in §27.5), we can state the intuitively compelling principle of *universal instantiation* as follows:

From a universal generalization, we can infer any particular instance.

(b) This first principle enables us to argue *from* a universal generalization. Our second principle will allow us to argue *to* a universal generalization. To introduce it, we will work through an example. So consider this inference:

B Everyone likes pizza. Whoever likes pizza likes ice cream. So everyone likes ice cream.

Again, obviously valid! But how can we derive the conclusion from the premisses with an informal proof? The trick is to consider an arbitrary representative from the domain.

Regimenting the premisses and conclusion using informal quantifier prefixes, the idea is that we can argue like this:

(1) (Everyone x is such that) x likes pizza. (premiss)
(2) (Everyone x is such that) if x likes pizza, then x likes ice cream. (premiss)
Now pick any person in the domain, temporarily dub them 'Alex'. Then:
(3) Alex likes pizza (from 1)
(4) If Alex likes pizza, Alex likes ice cream (from 2)
(5) Alex likes ice cream. (from 3, 4)
But Alex was arbitrarily chosen, and we appeal to no special facts about them; so what we can deduce about them applies to everyone:
(6) (Everyone x is such that) x likes ice cream. (from 5)

Now, the final step here is not, repeat *not*, relying on the hopeless idea that whatever is true of some individual in a domain is true of everyone/everything. (Here's Veronika. She is – as it happens – a woman, Slovak, and plays the violin. Plainly, we can't infer that everyone is a female Slovak violinist.) Rather, the principle at stake is this 'second-level' one (compare §20.7(a), §21.3, §22.1(a)):

> Suppose, given some background assumptions, we can infer that an arbitrary representative member of a domain is F. Then, from the same background assumptions, we can infer that everything in the domain is F (where the conclusion is general, i.e. no longer mentions the arbitrary representative).

But when can we treat some individual as an arbitrary representative of the domain? When we rely on no special distinguishing facts about that individual. In other words, when that individual features in no premisses or additional assumptions – so we can only draw on general knowledge about the domain in establishing that the individual in question is F.

(c) Let's immediately take another example of this principle of *universal generalization* at work. Consider:

C No horses are green. All frogs are green. So no horses are frogs.

Regimenting the premisses and conclusion, recasting the 'no' propositions using prefixed universal quantifiers and negation, we can now argue as follows:

(1) (Everything x is such that) if x is a horse, x is not green. (premiss)
(2) (Everything x is such that) if x is a frog, then x is green. (premiss)
Now pick any thing in the domain, temporarily dub it 'Arb'. Then:
(3) If Arb is a horse, Arb is not green. (from 1)
(4) If Arb is a frog, Arb is green. (from 2)
(5) If Arb is a horse, Arb is not a frog. (from 3, 4)
But Arb was arbitrarily chosen, and we appeal to no special facts about it; so what we can deduce about it applies equally to anything:
(6) (Everything x is such that) if x is a horse, x is not a frog. (from 5)

We derive (3) and (4) using our first quantifier principle. The step taking us from (3) and (4) to (5) is just propositional reasoning. We then derive our universally generalized conclusion (6) using our second quantifier principle.

(d) There is an old worry that goes back to George Berkeley. Think of the mathematical practice of starting a proof with, e.g, 'Let Δ be an arbitrary triangle . . . '. Following Berkeley, someone might jump in to ask what an arbitrary triangle could possibly be – how can it possibly be, in his words, "neither equilateral, isosceles, nor scalene; but all and none of these at once"?

However, taking Δ to be an arbitrary representative triangle in an argument, or taking Alex to be an arbitrary representative person, etc., is *not* to conjure up a peculiar sort of being which lacks specific properties (or has contradictory properties). Rather, to repeat, it is just to consider an individual from the domain while not relying on anything distinctive about it.

31.2 Arguing with existential quantifiers

(a) Now we introduce two principles for arguing with 'some'. This time, it is the principle for arguing *to* a generalization which is the simple one. Consider, for example,

D Socrates is a wise man. Therefore some man is wise.

Regimenting the conclusion with a quantifier prefix, we can render the argument as follows:

(1) Socrates is a man and Socrates is wise. (premiss)
(2) (Something x is such that) x is a man and x is wise. (from 1)

The principle we have used here – call it *existential generalization* – is perfectly straightforward: what is true of any particular named individual is certainly true of *something* in a suitably inclusive domain. Using the notion of an instance, we can put it this way:

> We can infer an existential generalization from any particular instance of it.

Note too this similar argument:

E Narcissus loves himself. Therefore someone loves Narcissus.

Again, a regimented proof is simple:

(1) Narcissus loves Narcissus. (premiss)
(2) (Someone x is such that) x loves Narcissus. (from 1)

(1) is a particular instance of the existentially quantified (2), where we replace the variable with a name; and we can again infer the existentially quantified proposition from the instance. The point of this little example is to highlight that, when we generalize from a claim like (1) involving a name by using a prefixed 'some' quantifier, we do not have to replace *all* the occurrences of the name with a variable to get a valid inference using our principle.

(b) That was easy. But our second principle for arguing with 'some' quantifiers takes rather more explanation. Consider another obviously valid argument:

F Someone likes pizza. Whoever likes pizza likes ice cream. So someone likes ice cream.

How can we show that the conclusion follows from the premisses?

(i) As a warm-up exercise, pretend for a moment that there are only two people in the domain, Jack and Jill. In this situation, the first premiss is tantamount to a disjunction, either Jack likes pizza or Jill does. Then in a familiar way we can *argue by cases* from this disjunctive premiss. Suppose, to take the first case, Jack likes pizza; by our second premiss, he likes ice cream; hence someone likes ice cream. Suppose, to take the second case, Jill likes pizza; then she likes ice cream; hence again someone likes ice cream. Therefore, in either case, we can conclude someone likes ice cream. Since we are given that one or the other case holds, we can conclude outright that someone likes ice cream.

(ii) So far so good. But now we need to drop the pretence that there are just two people. In general, we can't run through everyone in the domain case by case (that's probably quite impractical, and anyway we might not know exactly who is in the domain). What to do?

We can still argue as if by cases by *arguing from a representative case*, using an arbitrary instance of our existential premiss. So suppose Alex is an arbitrary pizza-lover. Given our second premiss, it follows that Alex likes ice cream, and hence that *someone* likes ice cream. Now, this conclusion that someone likes ice cream doesn't depend on our choice of representative pizza-lover. Hence, since our first premiss tells us that there *is* at least one pizza-lover to choose, our desired conclusion in fact follows outright.

To bring out the argument in (ii) clearly, we can regiment it as follows, indenting the line of proof when we make a temporary supposition (and invoking two now familiar principles for reasoning with quantifiers at lines 4 and 6):

(1) (Someone x is such that) x likes pizza. (premiss)
(2) (Everyone x is such that) if x likes pizza, x likes ice cream. (premiss)

Pick an arbitrary person and dub them 'Alex'. And now suppose

(3) Alex likes pizza. (supposition)
(4) If Alex likes pizza, Alex likes ice cream. (from 2)
(5) Alex likes ice cream. (from 3, 4)
(6) (Someone x is such that) x likes ice cream. (from 5)

But our premiss (1) tells us that there is someone *who is like Alex in being, as supposed, a pizza-lover. Our interim conclusion (6) doesn't depend on who Alex is. So we can argue using our representative instance of (1) to infer outright:*

(7) (Someone x is such that) x likes ice cream. (from 1, 3–6)

What makes Alex count as an arbitrary representative in this context? Again, the fact that we rely on no distinguishing information about Alex (other than that they like pizza) – in other words, we only draw on general knowledge about the domain in making inferences from the supposition (3).

How shall we spell out the general inferential principle being used here, where we argue from a representative case? We can state it like this:

> Suppose, given some background assumptions, (i) we have or can infer an existential generalization, and (ii) we can infer the claim C from an arbitrary instance of that existential generalization (where C is independent of the particular choice of instance). Then, from the same background assumptions, we can infer C.

Note, then, that this is another 'second-level' inferential principle.

(c) This principle is the most intricate of all the inference rules we will be stating in this book. But the underlying idea is simple enough. Let's walk slowly through a second informal example, where we again argue from a representative case. Here, then, is another valid Aristotelian syllogism to consider:

G Some pets are dragons. Nothing pink is a dragon. So some pets aren't pink.

Regimenting the premisses using prefixed quantifiers, we can set out a deduction of the conclusion from the premisses like this:

(1) (Something x is such that) x is a pet and x is a dragon. (premiss)
(2) (Everything x is such that) if x is pink, x is not a dragon. (premiss)
Pick any arbitrary object (in the relevant domain) and dub it 'Arb'. And now suppose
(3) Arb is a pet and Arb is a dragon. (supp'n)
(4) If Arb is pink, Arb is not a dragon. (from 2)
(5) Arb is a pet and Arb is not pink. (from 3, 4)
(6) (Something x is such that) x is a pet and x is not pink. (from 5)
But our premiss (1) tells us that there is something *which is like Arb in being, as supposed, a pet dragon. Our interim conclusion (6) doesn't depend on which particular thing Arb is. So we can conclude outright*
(7) (Something x is such that) x is a pet and x is not pink. (from 1, 3–6)

The reasoning from (3) and (4) to (5) is elementary propositional reasoning. Step (6) involves our first rule for 'some'. The final step at (7) is another application of our new principle, about which more in the next chapter.

31.3 Summary

> The four inference rules briskly introduced in this chapter are tidied-up versions of principles we already use in everyday reasoning, now applied to 'some' and 'every' in the form of quantifier prefixes. Here they are, restated

in something more like our style for giving formal rules of inference (of course, any sensible choice of variable is allowed):

(1) Universal instantiation: given *(Everything x is such that) x is F*, we can derive any instance *n is F*.

(2) Universal generalization: given the instance *n is F*, we can derive *(Everything x is such that) x is F* – so long as (i) n is arbitrary enough, which means that in establishing *n is F* we rely on no special facts about n, and (ii) the conclusion doesn't mention n.

(3) Existential generalization: given the instance *n is F*, we can derive *(Something x is such that) x is F*.

(4) 'As if by cases': Given *(Something x is such that) x is F*, and given *a proof that the instance n is F implies the conclusion C*, then we can derive C outright – so long as n is arbitrary enough and C doesn't mention n.

We can be content with these somewhat rough versions for now. The next chapter will give formal versions, in particular further clarifying the 'arbitrariness' conditions in two of the rules.

Exercises 31

Regiment the premisses and conclusions of the following arguments using informal prefixed quantifiers and variables. Then, using just the four quantifier principles we have met plus propositional reasoning, give informal derivations in the style of this chapter to show that the arguments are valid:

(1) No whales are fish. So no fish are whales.
(2) All leptons have half-integer spin. All electrons are leptons. So all electrons have half-integer spin.
(3) Some chaotic attractors are not fractals. Every Cantor set is a fractal. Hence some chaotic attractors are not Cantor sets.
(4) Some philosophers are logicians. All logicians are rational people. No rational person is a flat-earther. Therefore some philosophers are not flat-earthers.
(5) If Jo can do the exercises, then everyone in the class can do the exercises. Mo is in the class, and can't do the exercises. So Jo can't do the exercises.
(6) All lions and tigers are dangerous. Leo is a lion. So Leo is dangerous.

Give informal derivations warranting these arguments too (more difficult!):

(7) Everyone loves logic. Hence it isn't the case that someone doesn't love logic.
(8) Any philosopher who is not a fool likes logic. There is a philosopher who isn't a fool. Therefore not every philosopher fails to like logic.
(9) If Angharad loves every Welsh speaker, she loves Bryn. Angharad doesn't love Bryn. So there is some Welsh speaker who isn't loved by Angharad.
(10) There is someone who loves everyone. Hence everyone is loved by someone or other.

32 QL proofs

We now introduce inference rules for the use of quantifiers in formal QL arguments. This is the crucial step in developing quantificational logic. However, our rules will now look familiar, because they are just formal versions of the four intuitively compelling rules we met in the last chapter.

32.1 Dummy names in QL languages

(a) We need to start, though, by asking: just what are expressions like 'Alex' doing in our informal proofs in the last chapter?

'Alex' there functions somewhat like the lawyer's 'John Roe' or 'Jane Doe'. It plainly isn't serving as an ordinary proper name which *already* has a fixed reference. On the other hand, it isn't a bound pronoun either. It is a sort of 'dummy name', or 'temporary name', or 'ambiguous name', or 'arbitrary name' – all of those labels are in use, although perhaps none is entirely happy. Another, colourless, label for the logical use is 'parameter'. We will mostly use the first and last of these labels.

When we turn to formalizing our quantifier proofs, then, we will want symbols to play the role of these dummy names or parameters. There is no one agreed policy for supplying such symbols. There are three alternatives on the market:

(A) We can use the same symbols for free-standing parameters as we already use for variables-bound-to-quantifiers (in conventional jargon, the same symbols can appear as both 'free variables' and 'bound variables').

(B) We can use the same type of symbols for dummy names as for proper names (with just some of the names getting a fixed denotation in a given language).

(C) We can use a third, distinctive, type of symbol for dummy names.

The first policy has historically been the most common one among logicians. It perhaps does conform best to the practice of mathematicians who casually move between using letters as dummy or temporary names ('Let m be the number of positive roots ...') and using letters of the same kind as quantified variables ('For any positive integer n, $(n+1)(n-1) = n^2 - 1$'), often leaving it to context to make it clear what the symbols are doing. However, when we go formal, overloading symbols like this can cause trouble unless we spell out careful rules for their double use.

The second policy can also be made to work with a bit of care. But once we have distinguished, at the semantic level, (i) the fixed-denotation names built into a particular language from (ii) its further supply of dummy names, why not highlight the distinction by using syntactically different symbols for the two styles of name?

The third policy, then, may be less economical but it does make it easier to keep track of what is going on. It conforms to the good Fregean principle of marking important differences of semantic role by using different styles of symbols. It has distinguished antecedents, particularly in the work of the great 1930s logician Gerhard Gentzen. So (C) is the policy we adopt in this book.

(b) Given our decision, then, we now need to augment our definition of the shared logical apparatus of each QL language. Thus:

> Every QL language, as well as having an unlimited supply of *variables*, starting
>
> $\mathsf{x}, \mathsf{y}, \mathsf{z}, \ldots,$
>
> also has an unlimited supply of *dummy names*, starting
>
> $\mathsf{a}, \mathsf{b}, \mathsf{c}, \ldots.$
>
> Like proper names – and unlike variables – dummy names are also *terms*.

We will later do better than giving open-ended lists. For now, however, the crucial point is that the dummy names, the proper names and the variables of a language are to be sharply distinguished from each other as symbols (basically by using letters from the beginning, middle, and end of the alphabet).

The *syntactic* rules for wff-building introduced in §28.2(b) and §28.6(b) can remain the same, since we stated them as applying to terms generally. A predicate followed by the right number of terms – now proper names and/or dummy names – is still an atomic wff. Hence the atomic wffs of QL_1 also include

$\mathsf{Fa}, \mathsf{Gb}, \mathsf{Lbn}, \mathsf{Mcc}, \mathsf{Ramc}, \ldots.$

However, expressions with variables dangling free, without an associated quantifier prefix, still don't count as wffs for us, given we have adopted policy (C).

As to the *semantic* role of dummy names, just think of them for now as temporarily dubbing objects selected from the domain.

(c) We will need some terminology to distinguish wffs with dummy names from those without. So, for future use, let's say:

> A QL wff without dummy names is a closed wff or a *sentence*.
>
> A wff with one or more dummy names in it will be called an open wff or an *auxiliary wff*.

Hence, for example,

$\exists\mathsf{y}\,\mathsf{Mmy},\ \exists\mathsf{y}\forall\mathsf{x}\,\mathsf{Lxy},\ \forall\mathsf{z}(\mathsf{Hz} \to \exists\mathsf{y}\,\mathsf{Lzy}),\ \forall\mathsf{x}(\mathsf{Lx} \to \exists\mathsf{y}(\mathsf{Fy} \land \mathsf{Rxyn})),$

are sample QL_1 sentences; while

$$\mathsf{Lmb},\ \exists\mathsf{y}\,\mathsf{May},\ (\mathsf{Ha} \to \exists\mathsf{y}\,\mathsf{Lay}),\ (\mathsf{La} \to \exists\mathsf{y}(\mathsf{Fy} \wedge \mathsf{Rbyn})),$$

are also wffs, but auxiliary ones as they contain dummy names.

It is the *sentences* of a given QL language which will get stable, fixed, interpretations. And when we consider formal QL arguments, we will primarily be interested in arguments from zero or more *sentences* as premisses to a *sentence* as final conclusion. Non-sentences with dummy names will mainly play the same role as the informal expressions with 'Alex' and 'Arb' in the last section, i.e. they will mainly be used in auxiliary steps in the middle of natural deduction proofs – hence our label, 'auxiliary wffs'.

(d) One comment. Swapping between the different policy options (A), (B), and (C) for handling parameters in arguments will change which expressions count as wffs by changing the allowed *non*-sentences. But the class of *sentences* – i.e. wffs without parameters/dummy names, however they are implemented – can stay constant across the options. And, as you would expect, *the logical relations between sentences will stay constant too.* Hence, although we *do* have to make a policy choice here about the handling of parameters, in the most important respect it is not a very significant choice.

32.2 Schematic notation, and instances of quantified wffs

Before proceeding, it will be useful to gather together some notational conventions, old and new, for the use of symbols in schemas (compare §28.6):

> δ (*delta*, the Greek *d*) will be used to stand in for some dummy name.
>
> τ stands in for a term, which now can be a proper name or a dummy name.
>
> ξ, as before, stands in for some variable.
>
> $\alpha(\delta)$ and $\alpha(\tau)$ stand in for wffs which involve one or more occurrences of, respectively, the dummy name δ and the term τ.
>
> $\forall\xi\alpha(\xi)$ and $\exists\xi\alpha(\xi)$ stand in for wffs which begin with a universal or existential quantifier involving the variable ξ, and where the expression $\alpha(\xi)$ completing the wff contains one or more occurrences of the same variable ξ.

Now, our summary versions of the four quantifier rules in the last chapter were all stated in terms of the notion of an instance of a quantified claim. So we will need a formal definition of that notion:

> Suppose $\forall\xi\alpha(\xi)$ or $\exists\xi\alpha(\xi)$ is a quantified wff, τ is a term, and $\alpha(\tau)$ is the result of replacing every occurrence of ξ in $\alpha(\xi)$ by τ. Then $\alpha(\tau)$ is a wff and an *instance* of the quantified wff.

For example, here are a few quantified QL_1 wffs with some of their instances:

$\forall x\, \mathsf{Lxn}$:	$\mathsf{Lmn}, \mathsf{Lnn}, \mathsf{Lan}, \ldots,$
$\exists z(\mathsf{Fz} \wedge \mathsf{Gz})$:	$(\mathsf{Fn} \wedge \mathsf{Gn}), (\mathsf{Fa} \wedge \mathsf{Ga}), (\mathsf{Fc} \wedge \mathsf{Gc}), \ldots,$
$\forall x(\mathsf{Fx} \to \exists y \mathsf{Lxy})$:	$(\mathsf{Fm} \to \exists y\mathsf{Lmy}), (\mathsf{Fn} \to \exists y\mathsf{Lny}), (\mathsf{Fb} \to \exists y\mathsf{Lby}), \ldots.$

While the following are *not* pairs of quantified wffs and their instances:

$\neg\forall x\, \mathsf{Fx}$:	$\neg\mathsf{Fm},$
$(\exists z\mathsf{Fz} \wedge \mathsf{Gn})$:	$(\mathsf{Fn} \wedge \mathsf{Gn}),$
$\forall x(\mathsf{Fx} \to \exists y\mathsf{Lxy})$:	$\forall x(\mathsf{Fx} \to \mathsf{Lxm}).$

It is only wffs of the form $\forall\xi\alpha(\xi)$ or $\exists\xi\alpha(\xi)$ with initial quantifier prefixes – i.e. wffs with a quantifier as the main logical operator – which count as having instances.

32.3 Inference rules for '$\forall$'

(a) After those preliminaries, we can at last turn to building a Fitch-style natural deduction system for proofs involving quantifiers. (This section and the next are pivotal: so take them slowly and carefully!)

The basic shape of proofs, the principles for column-shifting, etc., are just the same as for PL proofs. And we can re-adopt our inference rules for dealing with the sentential connectives, now applied to wffs of our QL languages. Now we add a pair of rules for arguing with universal quantifiers.

Recall the first informal principle of inference we met in §31.1: from a universally quantified wff, we can infer *any* instance. As noted, this is often called 'universal instantiation'. But we will prefer the term $\forall$-*Elimination* for the formal version, in line with our previous conventions for naming rules (the idea, recall, is that an elimination rule for a logical operator o allows us to infer *from* a wff with o as its main operator). Using the obvious short-form label, then, the formal rule is very simply this:

($\forall$E) Given $\forall\xi\alpha(\xi)$, we can derive any instance $\alpha(\tau)$.

Exactly as in PL proofs, the 'given' inputs for the application of this and other rules of inference must be *available* (in the sense of §20.6).

Using a language with a suitable glossary, we can render argument **A** from §31.1 in the last chapter as

A′ $\mathsf{Fm}, \forall x(\mathsf{Fx} \to \mathsf{Gx}) \therefore \mathsf{Gm}.$

And since we have the familiar rules for the connectives available, our sketched informal argument warranting the inference therefore becomes

(1)	Fm	(Prem)
(2)	$\forall x(\mathsf{Fx} \to \mathsf{Gx})$	(Prem)
(3)	$(\mathsf{Fm} \to \mathsf{Gm})$	($\forall$E 2)
(4)	Gm	(MP 1, 3)

We annotate the application of the new rule ($\forall$E) in the now predictable way.

(b) The second principle we met is the 'universal generalization' rule: what applies to an arbitrary representative in a domain applies to everything. We will prefer the label ∀-*Introduction* for the formal version. And the rule becomes this: if we have already derived a wff with a dummy name then – so long as this dummy name really can be treated as temporarily dubbing an arbitrary representative because the name features in no relevant assumptions – we can go on to derive its universal generalization from the same background assumptions (where the generalization no longer mentions that representative).

In §24.6(c) we defined the idea of the *live assumptions* for a wff in a proof – i.e. the earlier available premisses/suppositions plus the wff on that line if it is itself a premiss/supposition. Then, more formally, the rule we want is this:

> Given an available wff $\alpha(\delta)$, where the dummy name δ does not occur in any live assumption for that wff, then we can universally quantify on δ, and derive $\forall\xi\alpha(\xi)$.

But note, if we quantify on δ, then that dummy name will not appear in the resulting wff $\forall\xi\alpha(\xi)$ – see §28.6(b). Hence our rule is equivalent to the following official version (think about it!):

> (∀I) We can derive $\forall\xi\alpha(\xi)$, given an available instance $\alpha(\delta)$ – so long as (i) the dummy name δ doesn't occur in any live assumption for that instance, and (ii) δ doesn't occur in the conclusion $\forall\xi\alpha(\xi)$.

Let's remark again, the variable ξ can't already appear quantified in $\alpha(\delta)$ or else $\forall\xi\alpha(\xi)$ won't be a wff.

Take, then, the argument **B** from the last chapter. Rendered into a suitable ad hoc QL language, this becomes:

B′ $\forall x\, Fx, \forall x(Fx \to Gx) \therefore \forall x\, Gx$.

And, using our new rule, a formal proof warranting the inference here can proceed exactly like the informal argument we gave for **B**:

(1)	$\forall x\, Fx$	(Prem)
(2)	$\forall x(Fx \to Gx)$	(Prem)
(3)	Fa	(∀E 1)
(4)	$(Fa \to Ga)$	(∀E 2)
(5)	Ga	(MP 3, 4)
(6)	$\forall x\, Gx$	(∀I 5)

Our way of annotating the inference to line (6) is as you would expect.

Similarly, we can formally render the argument **C** into a QL language like this, using an obvious glossary:

C′ $\forall x(Fx \to \neg Gx), \forall x(Hx \to Gx) \therefore \forall x(Fx \to \neg Hx)$.

Again, we can mirror our informal derivation with a formal one (for now, we will laboriously fill in the propositional reasoning between lines (4) and (11) below):

(1)	$\forall x(Fx \to \neg Gx)$	(Prem)
(2)	$\forall x(Hx \to Gx)$	(Prem)
(3)	$(Fa \to \neg Ga)$	(∀E 1)
(4)	$(Ha \to Ga)$	(∀E 2)
(5)	$\quad Fa$	(Supp)
(6)	$\quad \neg Ga$	(MP 5, 3)
(7)	$\quad\quad Ha$	(Supp)
(8)	$\quad\quad Ga$	(MP 7, 4)
(9)	$\quad\quad \bot$	(Abs 8, 6)
(10)	$\quad \neg Ha$	(RAA 7–9)
(11)	$(Fa \to \neg Ha)$	(CP 5–10)
(12)	$\forall x(Fx \to \neg Hx)$	(∀I 11)

Note that the dummy name 'a' does appear in the supposition at line (5), and appears again in another supposition at line (7). However – and this is crucial – by the time we get to line (11) both these suppositions have been discharged. Hence at line (11) *there are no live assumptions involving that dummy name.* So we *are* now allowed to universally quantify on it using (∀I).

32.4 Inference rules for '∃'

(a) Recall the PL rules. Think of the conjuncts of a conjunction as 'instances' of the conjunction; think too of the disjuncts of a disjunction as 'instances' of the disjunction. Then the (∧E) rule tells us we can infer any instance from a conjunction; and the (∨I) rule allows us to infer a disjunction from any instance.

Now a universal quantification is like a big conjunction (see §28.9), and the (∀E) elimination rule says that we can infer any instance from a universally quantified wff. Similarly, an existential quantification is like a big disjunction. And as we would then expect, the (∃I) introduction rule will say:

> (∃I) We can derive the wff $\exists\xi\alpha(\xi)$ from any given instance $\alpha(\tau)$.

This is our old 'existential generalization' principle from §31.2 in the last chapter: if a named individual satisfies a certain condition, we can infer the corresponding claim that *something* satisfies that condition.

Our formal rule can now be used in the following two intuitively correct mini-proofs, which track the informal proofs for **D** and **E** from §31.2:

(1)	$(Fn \wedge Gn)$	(Prem)
(2)	$\exists x(Fx \wedge Gx)$	(∃I 1)

and

(1)	Lnn	(Prem)
(2)	$\exists x\, Lxn$	(∃I 1)

(b) As before, the other existential quantifier rule – permitting us to argue 'as if by cases' – takes a bit more care to state. Here is the key idea again:

> We are given that something or other is F. Pick an arbitrary representative a from the domain and suppose that a is F. If this supposition implies a conclusion C which is independent of our choice of representative a, then – since we are given that something is F – we can conclude C.

So there are two conditions for an application of this rule which we need to capture. (i) We need a to be an arbitrary representative – i.e. we can't invoke any distinguishing facts about a. (ii) The conclusion C needs to be independent of our choice of representative – so C mustn't mention a.

For the formal rule, we prefer the label ∃-*Elimination* (compare ∨-Elimination), and here is a version of the rule we want:

(∃E) Given $\exists\xi\alpha(\xi)$, and a finished subproof from the instance $\alpha(\delta)$ as supposition to the conclusion γ – where (i) the dummy name δ is new to the proof, and (ii) γ does not contain δ – then we can derive γ.

Strictly speaking, we could allow occurrences of δ in earlier, but now completed, subproofs. However, our 'always use a new dummy name' condition (i) is memorable and is the simplest way of ensuring that δ features in no prior assumptions (nor in the relevant earlier wff $\exists\xi\alpha(\xi)$).

Here then is last chapter's informal argument **F** once more rendered into a suitable QL language, and now warranted using our new rule:

F′ $\exists x\,Fx,\ \forall x(Fx \to Gx)\ \therefore\ \exists x\,Gx.$

(1)	∃x Fx	(Prem)
(2)	∀x(Fx → Gx)	(Prem)
(3)	Fa	(Supp)
(4)	(Fa → Ga)	(∀E 2)
(5)	Ga	(MP 3, 4)
(6)	∃x Gx	(∃I 5)
(7)	∃x Gx	(∃E 1, 3–6)

We annotate the final step by giving the line number of the existentially quantified wff which we are invoking, and then noting the extent of the needed subproof from an instance of that wff with a new dummy name to the desired conclusion. Check that this step does obey our stated rule.

By the way, don't be surprised that an application of the existential quantifier *elimination* rule ends up proving an existentially quantified wff at line (7)! Look again: the elimination rule in our example is being applied to the *earlier* existentially quantified wff at line (1) – it is *that* existential wff which we are arguing from, using the (∃E) rule.

Similarly, we can formalize last chapter's syllogistic argument **G** like this:

G′ $\exists x(Fx \land Gx), \forall x(Hx \to \neg Gx) \therefore \exists x(Fx \land \neg Hx)$.

And here is a full-dress Fitch-style proof using our rules:

(1)	$\exists x(Fx \land Gx)$	(Prem)
(2)	$\forall x(Hx \to \neg Gx)$	(Prem)
(3)	$\quad (Fa \land Ga)$	(Supp)
(4)	$\quad (Ha \to \neg Ga)$	(∀E 2)
(5)	$\quad Fa$	(∧E 3)
(6)	$\quad Ga$	(∧E 3)
(7)	$\quad\quad Ha$	(Supp)
(8)	$\quad\quad \neg Ga$	(MP 7, 4)
(9)	$\quad\quad \bot$	(Abs 6, 8)
(10)	$\quad \neg Ha$	(RAA 7–9)
(11)	$\quad (Fa \land \neg Ha)$	(∧I 5, 10)
(12)	$\quad \exists x(Fx \land \neg Hx)$	(∃I 11)
(13)	$\exists x(Fx \land \neg Hx)$	(∃E 1, 3–12)

(c) It is, however, getting tedious to fill in routine derivations using rules for the sentential connectives. When we can argue from some previous QL wffs to a new wff by using propositional reasoning, why don't we now allow ourselves to jump straight to that new wff? For example, here is a briefer version of the last proof – we use 'PL' on the right to signal some PL-style inference:

(1)	$\exists x(Fx \land Gx)$	(Prem)
(2)	$\forall x(Hx \to \neg Gx)$	(Prem)
(3)	$\quad (Fa \land Ga)$	(Supp)
(4)	$\quad (Ha \to \neg Ga)$	(∀E 2)
(5)	$\quad (Fa \land \neg Ha)$	(PL 3, 4)
(6)	$\quad \exists x(Fx \land \neg Hx)$	(∃I 5)
(7)	$\exists x(Fx \land \neg Hx)$	(∃E 1, 3–6)

This version is not just shorter, but it highlights the quantificational part of our reasoning which is what we now care about. From now on, we will occasionally cut ourselves this degree of slack by skipping over PL reasoning when writing out Fitch-style proofs for arguments using quantifiers.

(d) For another example, consider the argument:

H Only logicians are wise. Some philosophers are not logicians. All who love Aristotle are wise. Hence some of those who don't love Aristotle are still philosophers.

This is valid (think about it!). Rendered into QL_1 this becomes

H′ $\forall x(Hx \to Gx), \exists x(Fx \land \neg Gx), \forall x(Lxo \to Hx) \therefore \exists x(\neg Lxo \land Fx)$.

A formal derivation to warrant the inference will therefore start

(1)	$\forall x(Hx \to Gx)$	(Prem)
(2)	$\exists x(Fx \land \neg Gx)$	(Prem)
(3)	$\forall x(Lxo \to Hx)$	(Prem)

And now a general point. When premisses are a mix of universal and existential quantifications, it is a good policy to instantiate the existential(s) first. Why? Because when we use (∃E), we must always begin by instantiating the existential quantification using a dummy name *new* to the proof. So you will need to get that name into play *before* you can later use it to instantiate some relevant universal quantification.

So let's instantiate (2) with a new dummy name, and then apply (∀E) to (1) and (3):

(4)	$(Fa \land \neg Ga)$	(Supp)
(5)	$(Ha \to Ga)$	(∀E 1)
(6)	$(Lao \to Ha)$	(∀E 3)

Now, our target conclusion is $\exists x(\neg Lxo \land Fx)$, which we can derive by existentially quantifying on the dummy name in $(\neg Lao \land Fa)$. But *that* wff follows from what we already have by propositional reasoning (how? – fill in the proof using PL rules!). Cutting ourselves some slack and skipping the details, we can conclude:

(7)	$(\neg Lao \land Fa)$	(PL 4, 5, 6)
(8)	$\exists x(\neg Lxo \land Fx)$	(∃I 7)
(9)	$\exists x(\neg Lxo \land Fx)$	(∃E 2, 4–8)

(e) For one more initial illustration of the existential rules in action, let's give a derivation for the obviously correct inference

I $(\exists x\, Fx \lor \exists x\, Gx)\ \therefore\ \exists x(Fx \lor Gx)$

The premiss is a disjunction, so we know that the overall shape of our proof is going to be an argument by cases, like this:

$(\exists x\, Fx \lor \exists x\, Gx)$	(Prem)
$\exists x\, Fx$	(Supp)
$\vdots$	
$\exists x(Fx \lor Gx)$	
$\exists x\, Gx$	(Supp)
$\vdots$	
$\exists x(Fx \lor Gx)$	
$\exists x(Fx \lor Gx)$	(∨E)

How do we fill in the dots? In each case, we have an existentially quantified supposition, and so we will expect to use the rule (∃E). So we will start a new subproof by instantiating the quantified wff – using a new dummy name – and aiming for the conclusion $\exists x(Fx \lor Gx)$. Here's a completed proof:

(1)	$(\exists x\,Fx \lor \exists x\,Gx)$	(Prem)
(2)	$\quad \exists x\,Fx$	(Supp)
(3)	$\quad\quad Fa$	(Supp)
(4)	$\quad\quad (Fa \lor Ga)$	($\lor$I 3)
(5)	$\quad\quad \exists x(Fx \lor Gx)$	($\exists$I 4)
(6)	$\quad \exists x(Fx \lor Gx)$	($\exists$E 2, 3–5)
(7)	$\quad \exists x\,Gx$	(Supp)
(8)	$\quad\quad Gb$	(Supp)
(9)	$\quad\quad (Fb \lor Gb)$	($\lor$I 8)
(10)	$\quad\quad \exists x(Fx \lor Gx)$	($\exists$I 9)
(11)	$\quad \exists x(Fx \lor Gx)$	($\exists$E 7, 8–10)
(12)	$\exists x(Fx \lor Gx)$	($\lor$E 1, 2–6, 7–11)

32.5 Quantifier equivalences

We now have introduction and elimination rules for the two QL quantifiers. The good news is that *these are all the quantifier rules we need*. And if we allow ourselves to jump over merely propositional reasoning, formal quantifier proofs using these rules can often be pretty neat and natural.

We will work through more examples in the next chapter. But first, it is helpful to show that our rules are enough to prove the following fundamental facts:

(a) The choice of the variable we use when we quantify doesn't matter (so long as it is new to the wff).

(b) An initial quantifier prefix of the form $\forall\xi$ is equivalent to the corresponding $\neg\exists\xi\neg$, and $\exists\xi$ is equivalent to $\neg\forall\xi\neg$.

(c) Two adjacent initial quantifiers of the same flavour can be interchanged, and the result will be provably equivalent.

(a) First then, we warrant the trivial inference $\forall x\,Fx \mathrel{.\,\dot{.}\,.} \forall y\,Fy$:

(1)	$\forall x\,Fx$	(Prem)
(2)	Fa	($\forall$E 1)
(3)	$\forall y\,Fy$	($\forall$I 2)

The reverse inference is of course proved the same way, showing that the wffs are equivalent. And this simple pattern of proof generalizes. For any wffs $\forall\xi\alpha(\xi)$ and $\forall\zeta\alpha(\zeta)$, where the second results from the first by swapping all occurrences of the variable ξ for a variable ζ new to the expression, there will be a derivation of the shape:

(1)	$\forall\xi\alpha(\xi)$	(Prem)
(2)	$\alpha(\delta)$	($\forall$E 1)
(3)	$\forall\zeta\alpha(\zeta)$	($\forall$I 2)

We can also warrant the trivial inference $\exists x\,Fx \mathrel{.\,\dot{.}\,.} \exists y\,Fy$ as follows:

$$\begin{array}{lll}
(1) & \exists x\, Fx & \text{(Prem)} \\
(2) & \quad Fa & \text{(Supp)} \\
(3) & \quad \exists y\, Fy & (\exists\text{I } 2) \\
(4) & \exists y\, Fy & (\exists\text{E } 1,\ 2\text{–}3)
\end{array}$$

Again the pattern of proof generalizes in the obvious way.

(b) Next, we want to warrant the inference $\forall x\, Fx \therefore \neg\exists x\neg Fx$. We need to derive the negation of $\exists x\neg Fx$, so we assume that wff at (2) and aim for absurdity. And how can we now use this new existential assumption except by using ($\exists$E)? – so we suppose an instance at (3) and continue by aiming for absurdity:

$$\begin{array}{lll}
(1) & \forall x\, Fx & \text{(Prem)} \\
(2) & \quad \exists x\neg Fx & \text{(Supp)} \\
(3) & \quad\quad \neg Fa & \text{(Supp)} \\
(4) & \quad\quad Fa & (\forall\text{E } 1) \\
(5) & \quad\quad \bot & \text{(Abs 4, 3)} \\
(6) & \quad \bot & (\exists\text{E } 2,\ 3\text{–}5) \\
(7) & \neg\exists x\neg Fx & \text{(RAA 2–6)}
\end{array}$$

Since the absurdity sign doesn't contain the dummy name 'a', the application of ($\exists$E) conforms to the stated rule.

How do we warrant the converse inference $\neg\exists x\neg Fx \therefore \forall x\, Fx$? To get the conclusion, we can try to prove Fa and then universally generalize. So how can we prove Fa? Let's try that frequently useful dodge: assume the opposite and aim for a reductio!

$$\begin{array}{lll}
(1) & \neg\exists x\neg Fx & \text{(Prem)} \\
(2) & \quad \neg Fa & \text{(Supp)} \\
(3) & \quad \exists x\neg Fx & (\exists\text{I } 2) \\
(4) & \quad \bot & \text{(Abs 3, 1)} \\
(5) & \neg\neg Fa & \text{(RAA 2–4)} \\
(6) & Fa & \text{(DN 5)} \\
(7) & \forall x\, Fx & (\forall\text{I } 6)
\end{array}$$

Note, the final step is again legal, since the dummy name 'a' in (6) occurs in no live assumption, i.e. no premiss or undischarged assumption.

Now for a pair of proofs showing that we can derive $\exists x\, Fx$ and $\neg\forall x\neg Fx$ from each other. First, we want a proof of the following shape:

$$\begin{array}{ll}
\exists x\, Fx & \text{(Prem)} \\
\quad Fa & \text{(Supp)} \\
\quad \vdots & \\
\quad \neg\forall x\neg Fx & \\
\neg\forall x\neg Fx & (\exists\text{E})
\end{array}$$

Since the conclusion of the subproof begins with a negation, the obvious thing to do is assume $\forall x \neg Fx$ and aim for a reductio. So we can fill in the dots like this:

(1)	$\mid \exists x\, Fx$	(Prem)
(2)	$\mid \quad \mid Fa$	(Supp)
(3)	$\mid \quad \mid \quad \mid \forall x \neg Fx$	(Supp)
(4)	$\mid \quad \mid \quad \mid \neg Fa$	($\forall$E 3)
(5)	$\mid \quad \mid \quad \mid \bot$	(Abs 2, 4)
(6)	$\mid \quad \mid \neg\forall x \neg Fx$	(RAA 3–5
(7)	$\mid \neg\forall x \neg Fx$	($\exists$E 1, 2–6)

The derivation in the reverse direction is rather more interesting. Our premiss is $\neg\forall x \neg Fx$; how are we going to derive the desired conclusion $\exists x\, Fx$? Again, what else can we do but assume the opposite and aim for a reductio? So we have:

$\mid \neg\forall x \neg Fx$	(Prem)
$\mid \quad \mid \neg\exists x Fx$	(Supp)
$\mid \quad \mid \vdots$	
$\mid \quad \mid \bot$	
$\mid \neg\neg\exists x Fx$	(RAA)
$\mid \exists x\, Fx$	(DN)

To derive $\bot$ we'll want to prove $\forall x \neg Fx$. How can we do that? By generalizing something like $\neg Fa$. Let's try to prove *that* by supposing Fa and aiming for another reductio:

(1)	$\mid \neg\forall x \neg Fx$	(Prem)
(2)	$\mid \quad \mid \neg\exists x Fx$	(Supp)
(3)	$\mid \quad \mid \quad \mid Fa$	(Supp)
(4)	$\mid \quad \mid \quad \mid \exists x\, Fx$	($\exists$I 3)
(5)	$\mid \quad \mid \quad \mid \bot$	(Abs 4, 2)
(6)	$\mid \quad \mid \neg Fa$	(RAA 3–5)
(7)	$\mid \quad \mid \forall x \neg Fx$	($\forall$I 6)
(8)	$\mid \quad \mid \bot$	(Abs 7, 1)
(9)	$\mid \neg\neg\exists x Fx$	(RAA 2–8)
(10)	$\mid \exists x\, Fx$	(DN 9)

(c) Finally, here are a couple of simpler proofs illustrating how we can swap the order of two initial quantifiers of the same flavour. First, universals:

(1)	$\mid \forall x \forall y\, Lxy$	(Prem)
(2)	$\mid \forall y\, Lay$	($\forall$E 1)
(3)	$\mid Lab$	($\forall$E 2)
(4)	$\mid \forall x\, Lxb$	($\forall$I 3)
(5)	$\mid \forall y \forall x\, Lxy$	($\forall$I 4)

Next, let's warrant the inference $\exists x \exists y\, Lxy \therefore \exists y \exists x\, Lxy$. Given the existential premiss, we need to instantiate it in a supposition to prepare for an argument by ($\exists$E). But this instance is another existentially quantified wff, so we need next to take an instance of *that*. Following this plan, we get:

(1)	$\exists x \exists y\, Lxy$	(Prem)
(2)	$\quad \exists y\, Lay$	(Supp)
(3)	$\qquad Lab$	(Supp)
(4)	$\qquad \exists x\, Lxb$	($\exists$I 3)
(5)	$\qquad \exists y \exists x\, Lxy$	($\exists$I 4)
(6)	$\quad \exists y \exists x\, Lxy$	($\exists$E 2, 3–5)
(7)	$\exists y \exists x\, Lxy$	($\exists$E 1, 2–6)

Do pause to double-check that the applications of ($\forall$I) and ($\exists$E) in our last two proofs do obey the official rules!

(d) There is, of course, nothing special about any of our particular examples in this section. To take the last case, not only does $\exists x \exists y\, Lxy$ imply $\exists y \exists x\, Lxy$, but the same line of argument shows that, for any variables ξ and ζ, a wff of the form $\exists\xi\exists\zeta\varphi$ implies the corresponding wff with the quantifiers swapped around, $\exists\zeta\exists\xi\varphi$. Similarly, the other results in this section can be generalized.

We could therefore use these generalized results, if we want, to augment our QL proof system with *derived rules* in the sense of §23.2. For example, we could add a rule allowing change of variables as in (a). Or a rule allowing ourselves to swap around initial quantifiers of the same flavour as in (c). But probably the most useful short-cut rules to add would be this pair, generalizing (b):

($\forall\exists$) Given one of $\forall\xi\alpha(\xi)$ and $\neg\exists\xi\neg\alpha(\xi)$, you can derive the other.

($\exists\forall$) Given one of $\exists\xi\alpha(\xi)$ and $\neg\forall\xi\neg\alpha(\xi)$, you can derive the other.

However, we won't officially adopt these additional rules (though feel free to use them if you want). For remember, our principal aim is to develop an *understanding* of the core principles of our proof system. Our focus now is on understanding the quantificational part of the system (that's why we will be happy occasionally to skip over the details of PL reasoning in the middle of proofs, so as not to obscure a proof's quantificational core). But, as before, we are not particularly concerned to construct ever more complicated proofs for which derived rules might come in handy as short-cuts.

32.6 QL theorems

(a) So far, all of our examples of QL formal derivations start from sentences as premisses and end up with sentences as final conclusions (even if the derivations typically proceed via auxiliary wffs which use dummy names). Now for proofs which start from *no* premisses to reach some sentence as conclusion.

Let's extend some earlier terminology from §23.1:

A sentence which concludes a QL proof from zero premisses is a *theorem*.

Rather unexcitingly, QL sentences which are instances of tautological schemas will be QL theorems (since the propositional rules are still in play). And equally unexcitingly, whenever we have a QL proof from α to γ (where these are sentences), there will be a corresponding theorem of the form $(\alpha \to \gamma)$. For example, the following is a theorem:

J $\quad ((\exists x\, Fx \lor \exists x\, Gx) \to \exists x(Fx \lor Gx))$.

Just suppose the antecedent, derive the consequent as in our proof for **I**, and apply the conditional proof principle (CP).

(b) For a still rather unexciting example of a different kind, consider

K $\quad \forall x((Fx \land Hx) \to Fx)$,

which in QL_1 expresses the logical truth that whoever is a wise philosopher is indeed a philosopher. This certainly *ought* to be demonstrable just from the rules governing the logical operators. And of course it is:

(1)		
(2)	$\quad\quad (Fa \land Ha)$	(Supp)
(3)	$\quad\quad Fa$	(∀E 2)
(4)	$((Fa \land Ha) \to Fa)$	(CP 2–3)
(5)	$\forall x((Fx \land Hx) \to Fx)$	(∀I 4)

Check that this *is* a correctly formed proof.

(c) Here is a much more interesting example to finish the chapter. The following is another QL theorem (in any language containing a binary predicate 'L'):

$$\neg\exists x\forall y(Lxy \leftrightarrow \neg Lyy).$$

Or, rather, unpacking our occasionally useful double-arrow shorthand for the biconditional, *this* is a theorem:

L $\quad \neg\exists x\forall y((Lxy \to \neg Lyy) \land (\neg Lyy \to Lxy))$.

If we suppose this wff without its initial negation sign to be true, then we can very quickly derive absurdity:

(1)		
(2)	$\quad\quad \exists x\forall y((Lxy \to \neg Lyy) \land (\neg Lyy \to Lxy))$	(Prem)
(3)	$\quad\quad\quad\quad \forall y((Lay \to \neg Lyy) \land (\neg Lyy \to Lay))$	(Supp)
(4)	$\quad\quad\quad\quad ((Laa \to \neg Laa) \land (\neg Laa \to Laa))$	(∀E 3)
(5)	$\quad\quad\quad\quad \bot$	(PL 4)
(6)	$\quad\quad \bot$	(∃E 2, 3–5)
(7)	$\neg\exists x\forall y((Lxy \to \neg Lyy) \land (\neg Lyy \to Lxy))$	(RAA 2–6)

How does this proof work? We start our subproof at (3) with an instance of the existential quantification at (2), with a view to using (∃E). We can then instantiate the quantified wff (3) with *any* term, including the term 'a' already in the wff, to get (4). But (4) tells us that Laa is true if and only it is false. And that is impossible, whatever we might be using 'a' as a dummy name for. Unsurprisingly, there is a simple PL proof of absurdity from (4) – find it! Finally, (∃E) can then be correctly applied to derive (6), and we can then apply (RAA).

Now suppose that we interpret the predicate 'L' as ① *shaves* ②, and take the domain to be *the men in some particular village*. Then **L** tells us that there is no man in the village who shaves all and only those who do not shave themselves. Sometimes this is called the Barber Paradox – but there is no genuine paradox here, just a logical theorem that there can be no such person!

Suppose instead that we interpret 'L' as ① *is a member of* ②, and take the domain to be the universe of *sets* (i.e. sets as mathematicians conceive them, treated as objects in their own right – compare §25.5). Then **L** tells us that there is no set which has as its members just those sets which are not members of themselves. Think of a set which doesn't contain itself as a *normal* set: then we have shown that there is no set of all normal sets. This is, famously, Russell's Paradox. And this time the label 'paradox' is perhaps more appropriate.

For, assuming we can understand the mathematicians' usual idea of a set as an object over and above its members, '① is a normal set' seems a perfectly sensible unary predicate. And it is a rather plausible principle that, given a sensible unary predicate, we can gather together the things that satisfy the predicate into a set. So it is a surprise to find, purely as a matter of logic, that there can be no set of normal sets – our plausible principle can't be applied across the board.

(According to most set-theorists, by the way, all genuine sets are normal – the thought being that a set newly gathers together objects that have in some sense 'already' been formed, so a set can't contain itself. On this view, our argument shows that there is no universal set of all sets.)

32.7 Summary

To formalize quantifier arguments, we need symbols in our QL languages available for use as dummy names. We take the option of using special-purpose symbols, lower-case letters from the beginning of the alphabet. Syntactically, these dummy names behave the same way as proper names, providing more terms.

Guided by the informal rules summarized at the end of the previous chapter, we presented pairs of introduction and elimination rules for the formal universal and existential quantifiers. (We will not restate the formal rules now, but instead we will set them out diagrammatically at the beginning of the next chapter.)

Using these rules, we have found proofs for some simple syllogistic arguments, and also demonstrated some basic quantifier equivalences – we can swap variables, swap the order of quantifiers of the same kind, and swap $\forall\xi$

for $\neg\exists\xi\neg$, etc.

Exercises 32

(a) Revisit Exercises 31, and render the informal arguments given there into suitable QL languages, and then provide formal derivations of the conclusions from the premisses.

(b) Also translate the following into suitable QL languages, and again provide formal derivations of the conclusions from the premisses:

(1) Everyone is such that, if they admire Ludwig, then the world has gone mad. Therefore, if someone admires Ludwig, the world has gone mad.
(2) If Jones is a bad philosopher, then some Welsh speaker is irrational; but every Welsh speaker is rational; hence Jones is not a bad philosopher.
(3) Jack is taller than Jill. Someone is taller than Jack. If a first person is taller than a second, and the second is taller than a third, then the first person is taller than the third. Hence someone is taller than both Jack and Jill.
(4) Every logician admires Gödel. Whoever admires someone is not without feeling. Hence no logician is without feeling.
(5) Either not everyone liked the cake or someone baked an excellent cake. If I'm right, then whoever bakes an excellent cake ought to be proud. So if everyone liked the cake and I'm right, then someone ought to be proud.

Also translate the following into suitable QL languages and show that they are logical truths by deriving them as theorems:

(6) Everyone is either a logician or not a logician.
(7) It's not the case that all logicians are wise while someone is an unwise logician.
(8) Everyone has someone whom either they love (despite that person loving themself!) or whom they don't love (despite that person *not* loving themself!).

(c*) Suppose we set up QL-style languages with only the universal quantifier built in. Expressions of the form $\exists\xi\alpha(\xi)$ are now introduced into such a language simply as abbreviations for corresponding expressions of the form $\neg\forall\xi\neg\alpha(\xi)$. Show that the familiar rules ($\exists$I) and ($\exists$E) would be derived rules of this system.

(d*) Now suppose we had set up a QL-style language with a 'no' quantifier as the sole type of quantifier, expressed using the quantifier-former 'N'; so a wff $\mathsf{N}\xi\alpha(\xi)$ holds when *nothing* satisfies the condition expressed by α. Show that the resulting language would be expressively equivalent to a standard two-quantifier QL languages. What would be appropriate introduction and elimination rules for this new quantifier?

(e*) When discussing pairs of PL introduction and elimination rules, we saw that they fitted together in a harmonious way, with the elimination rule as it were reversing or undoing an application of the introduction rule. Can something similar be said about the pairs of QL introduction and elimination rules?

33 More QL proofs

We continue to explore the Fitch-style system for arguing in QL languages. We start by setting out the new quantifier rules again, and stressing again the need for the restrictions on the dummy names used in applications of (∀I) and (∃E). Then we work through some more examples of quantifier proofs, commenting on various issues as we go.

33.1 The QL rules again

(a) Here are the introduction and elimination rules for the two quantifiers, now in diagrammatic form, arranged to bring out some relations between them:

Rules for quantifiers

$$(\forall\mathrm{E}) \quad \begin{array}{c} \forall\xi\alpha(\xi) \\ \vdots \\ \alpha(\tau) \end{array} \qquad\qquad (\exists\mathrm{I}) \quad \begin{array}{c} \alpha(\tau) \\ \vdots \\ \exists\xi\alpha(\xi) \end{array}$$

$$(\forall\mathrm{I}) \quad \begin{array}{c} \alpha(\delta) \\ \vdots \\ \forall\xi\alpha(\xi) \end{array} \qquad\qquad (\exists\mathrm{E}) \quad \begin{array}{l} \exists\xi\alpha(\xi) \\ \vdots \\ \quad \begin{array}{|l} \alpha(\delta) \\ \hline \vdots \\ \gamma \end{array} \\ \gamma \end{array}$$

In all these rules, $\alpha(\tau)$ or $\alpha(\delta)$ is an instance of the corresponding quantified wff; as usual τ can be any kind of term, but δ must be a dummy name.

The following restrictions must be observed on the dummy names δ:

For (∀I), δ must not appear in any live assumption for $\alpha(\delta)$ or in the conclusion $\forall\xi\alpha(\xi)$ (all occurrences of δ are replaced by ξ).
For (∃E), δ must be new to the proof and must not appear in the conclusion γ.

(b) The formal rules (∀E) and (∃I) correspond to obviously correct informal inferences between propositions.

The formal derivation rule (∃E) corresponds to a surely reliable informal inference between entailments, which we spelt out and called 'arguing as if by cases' in §31.2.

Finally, the fourth rule, (∀I) corresponds to another intuitively correct second-level principle, which we spelt out and called 'universal generalization' in §31.1). Modulated into a formal key, that informal principle has become:

> Suppose, given some background assumptions, we can validly infer $\alpha(\delta)$ (where the temporary name δ doesn't appear in those assumptions). Then from the same background, we can validly infer $\forall\xi\alpha(\xi)$.

But now compare (∀I) with (RAA), (∨E), (CP), and (∀E), the other second-level rules. We have set out applications of those rules by *indenting* the relevant stretch of argument that the rule appeals to. It would be consistent to do this again when applying (∀I). In other words, instead of laying out applications of the ∀-introduction rule as in ($\forall I_1$) below, we could instead use something like ($\forall I_2$). We start with a blank line heading a subproof (a blank line as we are making no new supposition), then δ is introduced in the subproof and we eventually derive the required wff $\alpha(\delta)$:

$$(\forall I_1) \quad \begin{array}{c} \alpha(\delta) \\ \vdots \\ \forall\xi\alpha(\xi) \end{array} \qquad\qquad (\forall I_2) \quad \begin{array}{l} \begin{array}{|l} \\ \hline \vdots \\ \alpha(\delta) \end{array} \\ \forall\xi\alpha(\xi) \end{array}$$

This second layout is exactly what we find in some standard Fitch-style systems, often with the subproof's vertical line being additionally tagged with the dummy name which is being introduced. For example, instead of our proof **B′** in §32.3, set out as on the left, we can find something like the version on the right:

$$\begin{array}{|l} \forall x\, Fx \\ \forall x(Fx \to Gx) \\ \hline Fa \\ (Fa \to Ga) \\ Ga \\ \forall x\, Gx \end{array} \qquad\qquad \begin{array}{|l} \forall x\, Fx \\ \forall x(Fx \to Gx) \\ \hline \quad a \begin{array}{|l} \\ \hline Fa \\ (Fa \to Ga) \\ Ga \end{array} \\ \forall x\, Gx \end{array}$$

But is the extra visual complication *really* a helpful addition, once we understand the principle underlying (∀I) proofs? This is a judgement call. We will prefer to stick to our simpler (and equally standard) style of layout.

33.2 How to misuse the QL rules

(a) The derivation rules (∀E) and (∃I) are entirely straightforward. The rules (∀I) and (∃E) require more care, however. In the last chapter, we motivated the crucial restrictions on the use of dummy names in these rules. It will help to fix ideas if we now consider some ways derivations can go disastrously wrong if we offend against those restrictions.

So first, let's give a fake derivation from $\exists x\, Fx$ to $\forall x\, Fx$:

(1)	$\exists x\, Fx$	(Prem)
(2)	$\quad Fa$	(Supp)
(3)	$\quad \forall x\, Fx$	(!!∀I 2)
(4)	$\forall x\, Fx$	(∃E 1, 2–3)

Intuitively, things go wrong at (3): from the supposition that some chosen individual is (say) a logician, we can't infer that everyone is! Formally, the hopeless move is blocked by the restriction that says we can't universally generalize on a dummy name if it features in a live assumption (an undischarged supposition).

Here's another 'proof' for the same bad inference:

(1)	$\exists x\, Fx$	(Prem)
(2)	$\quad Fa$	(Supp)
(3)	$\quad Fa$	(Iter 2)
(4)	Fa	(!!∃E 1, 2–3)
(5)	$\forall x\, Fx$	(∀I 4)

This time the gruesome mistake is at line (4). Our subproof doesn't reach a conclusion independent of our choice of instance (2) of the initial existential quantification. So we can't argue 'as if by cases'. Formally, what's gone wrong is that we have tried to apply (∃E) when the subproof ends with a wff which still contains the relevant dummy name introduced at the beginning of the subproof.

(b) Now let's 'prove' that, if everyone loves someone, then someone loves themself – so we will argue from $\forall x \exists y\, Lxy$ to $\exists x\, Lxx$:

(1)	$\forall x \exists y\, Lxy$	(Prem)
(2)	$\exists y\, Lay$	(∀E 1)
(3)	$\quad Laa$	(Supp)
(4)	$\quad \exists x\, Lxx$	(∃I 3)
(5)	$\exists x\, Lxx$	(!!∃E 2, 3–4)

What's gone wrong here? At line (2), we have in effect picked an individual a from the domain and we know from (1) that a loves *someone* or other. We can now go on chose a representative beloved for a at (3) – but we are not entitle to continue the argument by taking that representative to be a again!

Here's a similar case. We will 'prove' that, given first that someone is a logician and second that someone is irrational, it follows that there is an irrational logician.

Using the obvious glossary, here then is a fake derivation from $\exists x\, Fx$ and $\exists x\, Gx$ to $\exists x(Fx \wedge Gx)$:

(1)	$\exists x\, Fx$	(Prem)
(2)	$\exists x\, Gx$	(Prem)
(3)	$\quad Fa$	(Supp)
(4)	$\quad\quad Ga$	(Supp)
(5)	$\quad\quad (Fa \land Ga)$	($\land$I 3, 4)
(6)	$\quad\quad \exists x(Fx \land Gx)$	($\exists$I 5)
(7)	$\quad \exists x(Fx \land Gx)$	(!!$\exists$E 2, 4–6)
(8)	$\exists x(Fx \land Gx)$	($\exists$E 1, 3–7)

What's gone wrong this time? We've picked a representative logician a at line (3). Then we've picked a representative irrational person at line (4) – but we can't assume that this is a again.

Formally, in both these last two fake proofs, we have broken the rule which tells us that when we instantiate an existential quantification to start a subproof for ($\exists$E) we must always use a dummy name new to the proof.

(c) Let's consider another pair of inferences:

$\exists y \forall x Lxy \ \therefore\ \forall x \exists y Lxy$,
$\forall x \exists y Lxy \ \therefore\ \exists y \forall x Lxy$.

The first is valid – why? But the reverse second inference isn't – why?

Here then is a genuine proof for the first inference:

(1)	$\exists y \forall x\, Lxy$	(Prem)
(2)	$\quad \forall x Lxa$	(Supp)
(3)	$\quad Lba$	($\forall$E 2)
(4)	$\quad \exists y\, Lby$	($\exists$I 3)
(5)	$\quad \forall x \exists y\, Lxy$	($\forall$I 4)
(6)	$\forall x \exists y\, Lxy$	($\exists$E 1, 2–5)

Check that this proof is correctly formed.

And now here is a 'proof' of the reverse inference:

(1)	$\forall x \exists y\, Lxy$	(Prem)
(2)	$\exists y Lay$	($\forall$E 1)
(3)	$\quad Lab$	(Supp)
(4)	$\quad \forall x\, Lxb$	(!!$\forall$I 3)
(5)	$\quad \exists y \forall x\, Lxy$	($\exists$I 4)
(6)	$\exists y \forall x\, Lxy$	($\exists$E 1, 2–5)

Even though the dummy name 'a' was introduced as picking out an arbitrary individual at (2), we can't now universally generalize on it to get (4). Why? Because we've made a special assumption about that individual at line (3).

(d) Let's have one last fake proof. We will 'demonstrate' that if everyone loves themself then everyone loves everyone – i.e. we will argue from $\forall x\, Lxx$ to $\forall x \forall y\, Lxy$:

(1)	$\forall x\, Lxx$	(Prem)
(2)	Laa	($\forall$E 1)
(3)	$\forall y\, Lay$	(!!$\forall$I 2)
(4)	$\forall x \forall y\, Lxy$	($\forall$I 3)

We go wrong at (3). From the assumption that some arbitrarily selected individual *a* loves themself, we can't infer that *a* loves everyone. Formally, we have offended against the rule that when we universally generalize on a dummy name, we have to replace *all* the occurrences of the name with the relevant variable. Equivalently, when we derive a universal quantification from an instance with a dummy name, that name must not appear in the quantified wff.

The moral of all our examples? Obeying those restrictions on the use of dummy names in the rules ($\forall$I) and ($\exists$E) is *essential* if you are to construct valid proofs.

33.3 Old and new logic: three proofs

In this section we look at three kinds of arguments which 'traditional' logicians had trouble with, to show that such arguments can be dealt with easily by our QL proof system which incorporates Fregean insights.

(a) Take first the simple two-step argument

A Everyone loves themself. So Maldwyn loves Maldwyn. So someone loves Maldwyn.

Why does this cause trouble for a logic of generality rooted in the Aristotelian tradition? Consider the *second* inference step. To explain its validity, the traditional logician – relying on a subject/predicate analysis of propositions – needs to discern a common (unary) predicate in its two propositions, namely 'loves Maldwyn'. But then, if 'Maldwyn loves Maldwyn' involves just *that* predicate, exactly why does it follow from the premiss of the *first* inference – which involves the quite different predicate 'loves themself'?

Formalized into a suitable QL language, however, **A** becomes this:

(1)	$\forall x\, Lxx$	(Prem)
(2)	Lmm	($\forall$E 1)
(3)	$\exists x\, Lxm$	($\exists$I 2)

The predicate-first notation is a mere matter of style. What is crucial is, first, that the middle proposition is now analysed at (2) as involving a *binary* predicate 'L' with two slots to be filled. And second, Frege's great insight, we need to understand a quantifier as something that can be tied to different numbers of places in the same predicate. Then we can see how the very same predicate can feature in both (1) and (3).

(b) Consider next

B Either every wombat is a mammal or every wombat is a marsupial. Hence every wombat is either a mammal or a marsupial.

The main logical operator of the premiss is the disjunction. The main operator of the conclusion is the quantifier. So we need a logic that can simultaneously handle sentential connectives and quantifiers.

Ancient Stoic logic copes with some propositional reasoning; Aristotelian logic copes with some quantifier reasoning. Much later – to continue the cartoon history – George Boole in the early nineteenth century came up with a logical algebra which could be interpreted two ways. It could be read as being about 'and', 'or', and 'not' (that's why we still speak of Boolean connectives) or alternatively as being about Aristotelian syllogisms. But Boole's system still couldn't deal with both at the same time.

By contrast, our Frege-based treatment of the quantifiers combines with a logic for the connectives to allow us to argue like this (using the obvious glossary):

(1)	$(\forall x(Fx \to Gx) \lor \forall x(Fx \to Hx))$	(Prem)
(2)	$\quad \forall x(Fx \to Gx)$	(Supp)
(3)	$\quad (Fa \to Ga)$	($\forall$E 2)
(4)	$\quad (Fa \to (Ga \lor Ha))$	(PL 3)
(5)	$\quad \forall x(Fx \to (Gx \lor Hx))$	($\forall$I 4)
(6)	$\quad \forall x(Fx \to Hx)$	(Supp)
(7)	$\quad (Fa \to Ha)$	($\forall$E 6)
(8)	$\quad (Fa \to (Ga \lor Ha))$	(PL 7)
(9)	$\quad \forall x(Fx \to (Gx \lor Hx))$	($\forall$I 8)
(10)	$\forall x(Fx \to (Gx \lor Hx))$	($\exists$E 1, 2–5, 6–9)

(c) The most intractable problem for traditional logic, however, was dealing with arguments involving propositions with quantifiers embedded inside the scope of other quantifiers (as we would now put it). Consider for example

C Every horse is a mammal. Hence every horse's tail is a mammal's tail.

To bring out the multiple generality in the conclusion, read it as 'Everything which is the tail of some horse is the tail of some mammal'. The argument is then evidently valid. Medieval logicians in the Aristotelian tradition, for all their insightful ingenuity, struggled to cope. But there's a straightforward QL proof.

Let's adopt a QL language with the following glossary:

F: ① is a horse,
G: ① is a mammal,
T: ① is a tail belonging to ②;

and take the domain to be inclusive enough, perhaps all terrestial physical objects. Then we can render **C** as

C′ $\forall x(Fx \to Gx) \ \therefore\ \forall x(\exists y(Txy \land Fy) \to \exists y(Txy \land Gy))$.

Now, we will expect the conclusion to be derived by universal quantifier introduction from a wff involving some dummy name, like this:

$\forall x(Fx \to Gx)$	(Prem)
$\vdots$	
$(\exists y(Tay \land Fy) \to \exists y(Tay \land Gy))$	
$\forall x(\exists y(Txy \land Fy) \to \exists y(Txy \land Gy))$	($\forall$I)

The penultimate line is a conditional; so we will presumably want to assume its antecedent and then derive its consequent. And this antecedent is an existentially quantified wff; so the natural thing to do is to assume an instance of it, with a view to arguing by an application of ($\exists$E), giving a proof shaped like this:

$\forall x(Fx \to Gx)$	(Prem)
$\quad \exists y(Tay \land Fy)$	(Supp)
$\quad\quad (Tab \land Fb)$	(Supp)
$\quad\quad \vdots$	
$\quad\quad \exists y(Tay \land Gy)$	
$\quad \exists y(Tay \land Gy)$	($\exists$E)
$(\exists y(Tay \land Fy) \to \exists y(Tay \land Gy))$	(CP)
$\forall x(\exists y(Txy \land Fy) \to \exists y(Txy \land Gy))$	($\forall$I)

Do pause at this stage to double-check how the applications of the two rules ($\exists$E) and ($\forall$I) obey the restrictions on their relevant dummy names 'a' and 'b'.

So given that outline proof, we now simply need to join up the remaining dots:

(1)	$\forall x(Fx \to Gx)$	(Prem)
(2)	$\quad \exists y(Tay \land Fy)$	(Supp)
(3)	$\quad\quad (Tab \land Fb)$	(Supp)
(4)	$\quad\quad (Fb \to Gb)$	($\forall$E 1)
(5)	$\quad\quad (Tab \land Gb)$	(PL 3, 4)
(6)	$\quad\quad \exists y(Tay \land Gy)$	($\exists$I 5)
(7)	$\quad \exists y(Tay \land Gy)$	($\exists$E 2, 3–6)
(8)	$(\exists y(Tay \land Fy) \to \exists y(Tay \land Gy))$	(CP 2–7)
(9)	$\forall x(\exists y(Txy \land Fy) \to \exists y(Txy \land Gy))$	($\forall$I 8)

(d) A comment on our handling of the last example. In regimenting the informal argument **C**, we chose to use a language including the predicate

T: ① is a tail belonging to ②.

You might have expected to find instead, say, the following two predicates:

H: ① is a tail,
B: ① belongs to ②.

Then instead of Txy, you would expect to use (Hx ∧ Bxy), etc.

It could be said that our predicate T is unnatural. And if we also wanted to regiment other arguments about animals and their parts, it might well be that we'll need to have predicates like H and B separately available. However, in the present context, replacing occurrences of Txy by (Hx ∧ Bxy) etc. just produces unnecessary complexity. Writing out our proof that way would make *no* use of the internal structure of this complex expression; *so why introduce it*?

A general principle: when we construct an ad hoc language for regimenting some argument(s), it is on the whole good policy to dig down only just as far as we need. As Quine famously puts it,

> "A maxim of shallow analysis prevails: expose no more logical structure than seems useful for the deduction or other inquiry at hand."

33.4 Five more QL proofs

Pause for breath! – and then let's work through five more examples.

(a) In §4.5, we informally showed that the following argument is valid:

D No girl likes any unreconstructed sexist; Caroline is a girl who likes anyone who likes her; Henry likes Caroline; hence Henry is not an unreconstructed sexist.

Using a QL language with a suitable glossary, and rendering the 'no' premiss with a universal quantifier, we can translate the argument like this:

D′ ∀x(Gx → ∀y(Fy → ¬Lxy)), (Gm ∧ ∀x(Lxm → Lmx)), Lnm ∴ ¬Fn

To construct a formal derivation of the conclusion, the obvious thing to do is to assume Fn and aim for a contradiction. So think through the following proof:

(1)	∀x(Gx → ∀y(Fy → ¬Lxy))	(Prem)
(2)	(Gm ∧ ∀x(Lxm → Lmx))	(Prem)
(3)	Lnm	(Prem)
(4)	Fn	(Supp)
(5)	Gm	(∧E 2)
(6)	∀x(Lxm → Lmx)	(∧E 2)
(7)	(Gm → ∀y(Fy → ¬Lmy))	(∀E 1)
(8)	∀y(Fy → ¬Lmy)	(MP 5, 7)
(9)	(Fn → ¬Lmn)	(∀E 8)
(10)	¬Lmn	(MP 4, 9)
(11)	(Lnm → Lmn)	(∀E 6)
(12)	Lmn	(MP 3, 11)
(13)	⊥	(Abs 12, 10)
(14)	¬Fn	(RAA 4–13)

What have we done here after line (4)? We have first disassembled the conjunction at line (2), giving us lines (5) and (6). Then we have instantiated the universal quantification at line (1) with the name 'm' to get the conditional with antecedent Gm, setting up the modus ponens inference to get line (8). We can likewise instantiate the new universal quantification at (8) in a way that links up to Fn earlier in the proof, to get (9) and hence (10).

And *now* the end is in sight. We just need to bring out the evident inconsistency between (3), (6), and (10), and we are done.

(b) Next, we return to another silly argument that we have met before:

E Everyone loves a lover; Romeo loves Juliet; so everyone loves Juliet.

We read the first premiss as saying: whoever x might be, then if x is a lover (i.e. if x loves someone), then everyone loves x. So, adopting a language with the obvious glossary, we can regiment this argument as

E′ $\forall x(\exists y\, Lxy \to \forall y\, Lyx),\ Lrj\ \therefore\ \forall x\, Lxj$

And now let's replicate the informal multi-step proof we gave back in §4.1:

(1)	$\forall x(\exists y\, Lxy \to \forall y\, Lyx)$	(Prem)
(2)	Lrj	(Prem)
(3)	$(\exists y\, Lry \to \forall y\, Lyr)$	(∀E 1)
(4)	$\exists y\, Lry$	(∃I 2)
(5)	$\forall y\, Lyr$	(MP 4, 3)
(6)	Ljr	(∀E 5)
(7)	$(\exists y\, Ljy \to \forall y\, Lyj)$	(∀E 1)
(8)	$\exists y\, Ljy$	(∃I 6)
(9)	$\forall y\, Lyj$	(MP 8, 7)

Again, at each step in this proof so far, we just do the natural thing. At step (3) we will want to instantiate the quantified premiss, and the obvious option is to use the name 'r' to give us a conditional with an antecedent that connects with (2). That quickly takes us to (5), and what is the natural term to instantiate *that* with? Surely use the name 'j' this time. And so the proof continues . . . .

We haven't quite arrived at our target conclusion, however, as we regimented the conclusion of **E** using 'x', the first available variable for the quantification. Bother! We could retranslate that conclusion as $\forall y\, Lyj$: but that's a bit sneaky. So let's keep ourselves honest and finish up by using the variable-changing trick we met in §32.5:

(10)	Laj	(∀E 9)
(11)	$\forall x\, Lxj$	(∀I 10)

(c) For our next example, consider the following argument (we are talking, let's suppose, about the logic class and the questions on a particular test):

F If everyone in the class can answer *every* question, then some questions are too easy. So at least one person in the class is such that, if *they* can answer every question, then some questions are too easy.

That is valid. Let's show this by a QL proof.

We might be tempted to reflect all the surface structure of the premiss and conclusion, and come up with a QL translation along the following lines:

$$(\forall x(Cx \to \forall y(Qy \to Axy)) \to \exists z(Qz \land Ez)) \;\therefore\; \exists x((Cx \land \forall y(Qy \to Axy)) \to \exists z(Qz \land Ez))$$

Here the quantifier runs over some inclusive domain containing both people and questions, and we are using a QL language with a glossary like

C: ① is in the class,
Q: ① is a question,
E: ① is too easy,
A: ① can answer ②.

But note that the quantified structures of the subformulas $\forall y(Qy \to Axy)$ and $\exists z(Qz \land Ez)$ do no work at all in the argument. And since the other $\forall x/\exists x$ quantifiers are restricted to people in the class, we might as well take those people to make up the whole universe of discourse. So, following Quine's maxim of shallow analysis, let's instead take a simpler language with the following glossary:

F: ① can answer every question,
P: Some of the questions (in the test) are too easy.
Domain: people in the class

Remember, we allow QL languages to have propositional letters (see §28.3(c)). And now our translated argument looks *very* much more manageable!

F′ $(\forall x\, Fx \to P) \;\therefore\; \exists x(Fx \to P)$

We've seen inferences of this form in §29.6. Here's a formal derivation (not entirely easy, but why is each step the natural one to make?)

(1)	$(\forall x\, Fx \to P)$	(Prem)
(2)	$\quad \neg\exists x(Fx \to P)$	(Supp)
(3)	$\quad\quad (Fa \to P)$	(Supp)
(4)	$\quad\quad \exists x(Fx \to P)$	(∃I 3)
(5)	$\quad\quad \bot$	(Abs 4, 2)
(6)	$\quad \neg(Fa \to P)$	(RAA 3–5)
(7)	$\quad Fa$	(PL 6)
(8)	$\quad \neg P$	(PL 6)
(9)	$\quad \forall x\, Fx$	(∀I 7)
(10)	$\quad P$	(MP 9, 1)
(11)	$\quad \bot$	(Abs 10, 8)
(12)	$\neg\neg\exists x(Fx \to P)$	(RAA 2–11)
(13)	$\exists x(Fx \to P)$	(DN 12)

(d) Next, here is an argument we met in the very first chapter and then informally argued to be valid in §3.1:

G Some philosophy students admire all logicians. No philosophy student admires anyone irrational. So no logician is irrational.

We can translate the 'no' propositions using universal or existential quantifiers. Let's start by going the first way. Choosing formal predicate letters to match the English, we can then render the argument like this:

G′ $\exists x(Px \land \forall y(Ly \to Axy)),\ \forall x(Px \to \forall y(Iy \to \neg Axy))\ \therefore\ \forall x(Lx \to \neg Ix)$

Using the rule of thumb 'instantiate existential premisses first', the overall shape of the proof we are looking for is:

$\exists x(Px \land \forall y(Ly \to Axy))$	(Prem)
$\forall x(Px \to \forall y(Iy \to \neg Axy))$	(Prem)
$\quad (Pa \land \forall y(Ly \to Aay))$	(Supp)
$\quad \vdots$	
$\quad \forall x(Lx \to \neg Ix)$	
$\forall x(Lx \to \neg Ix)$	($\exists$E)

Unpack the conjuncts in the supposition. And then it is natural to instantiate the second premiss using the dummy name 'a'. Which gets us painlessly to

(1)	$\exists x(Px \land \forall y(Ly \to Axy))$	(Prem)
(2)	$\forall x(Px \to \forall y(Iy \to \neg Axy))$	(Prem)
(3)	$\quad (Pa \land \forall y(Ly \to Aay))$	(Supp)
(4)	$\quad Pa$	($\land$E 3)
(5)	$\quad \forall y(Ly \to Aay)$	($\land$E 3)
(6)	$\quad (Pa \to \forall y(Iy \land \neg Aay))$	($\forall$E 2)
(7)	$\quad \forall y(Iy \to \neg Aay)$	(MP 4, 6)

So the remaining task is to get from (5) and (7) to the target conclusion of the subproof, i.e. to derive $\forall x(Lx \to \neg Ix)$.

But this is relatively simple (compare **C′** in the previous chapter). We instantiate the two universal quantifiers (not with 'a' again because we want an arbitrary instance we can generalize on later). And then continue:

(8)	$\quad (Lb \to Aab)$	($\forall$E 5)
(9)	$\quad (Ib \to \neg Aab)$	($\forall$E 7)
(10)	$\quad (Lb \to \neg Ib)$	(PL 8, 9)
(11)	$\quad \forall x(Lx \to \neg Ix)$	($\forall$I 10)
(12)	$\forall x(Lx \to \neg Ix)$	($\exists$E 1, 3–11)

(e) For a last example in this chapter, let's consider how things go if we alternatively translate the 'no' propositions in **G** using negated existential quantifiers, as in

G″ $\exists x(Px \land \forall y(Ly \to Axy)),\ \neg\exists x\exists y(Px \land (Iy \land Axy))\ \therefore\ \neg\exists x(Lx \land Ix)$

We will talk through how to discover a proof.

(Though let's never lose sight of the point we first stressed in §20.5. The key thing, always, is to make sure you *understand the general principles* deployed in proofs. For us, as philosophers, proof-discovery remains secondary, even if sometimes instructive.)

In this case, given $\mathbf{G}''$'s negative conclusion, the obvious strategy is to assume $\exists x(Lx \land Ix)$ at line (3) and aim for absurdity, so we can use a reductio argument.

We will then have *two* existential assumptions in play, at lines (1) and (3): so instantiate them in turn at lines (4) and (5). We get two conjunctions as a result. So next disassemble those. And we reach the following stage:

(1)	$\mid\ \exists x(Px \land \forall y(Ly \to Axy))$	(Prem)
(2)	$\mid\ \neg\exists x\exists y(Px \land (Iy \land Axy))$	(Prem)
(3)	$\mid\ \mid\ \exists x(Lx \land Ix)$	(Supp)
(4)	$\mid\ \mid\ \mid\ (Pb \land \forall y(Ly \to Aby))$	(Supp)
(5)	$\mid\ \mid\ \mid\ \mid\ (La \land Ia)$	(Supp)
(6)	$\mid\ \mid\ \mid\ \mid\ Pb$	($\land$E 4)
(7)	$\mid\ \mid\ \mid\ \mid\ \forall y(Ly \to Aby)$	($\land$E 4)
(8)	$\mid\ \mid\ \mid\ \mid\ La$	($\land$E 5)
(9)	$\mid\ \mid\ \mid\ \mid\ Ia$	($\land$E 5)

What next? How do we derive $\perp$? Well, obviously we will need to use the quantified wff at (7). What shall we instantiate it with? The natural choice is 'a' as that will give us a conditional whose antecedent we will already have at line (8). So we can continue:

(10)	$\mid\ \mid\ \mid\ \mid\ (La \to Aba)$	($\forall$E 7)
(11)	$\mid\ \mid\ \mid\ \mid\ Aba$	(MP 8, 10)

And now the end is in sight. For we can now conjoin (11) with some earlier lines to form the conjunction

(12)	$\mid\ \mid\ \mid\ \mid\ (Pb \land (Ia \land Aba))$	(PL 6, 9, 11)

But this is flatly inconsistent with (2). Bring out the contradiction, and the derivation completes itself, with the Fitch-style layout beautifully keeping track of the ins and outs of the proof:

(13)	$\mid\ \mid\ \mid\ \mid\ \exists y(Pb \land (Iy \land Aby))$	($\exists$I 12)
(14)	$\mid\ \mid\ \mid\ \mid\ \exists x\exists y(Px \land (Iy \land Axy))$	($\exists$I 13)
(15)	$\mid\ \mid\ \mid\ \mid\ \perp$	(Abs 14, 2)
(16)	$\mid\ \mid\ \mid\ \perp$	($\exists$E 3, 5–15)
(17)	$\mid\ \mid\ \perp$	($\exists$E 1, 4–16)
(18)	$\mid\ \neg\exists x(Lx \land Ix)$	(RAA 3–17)

Which, guided by some pretty natural strategic ideas, was not *too* hard. Just check that the two applications of ($\exists$E) are correctly done.

33.5 Summary

We have restated the formal introduction and elimination quantifier rules in a diagrammatic form. We commented on our preferred (simpler) way of laying out (∀I) inferences.

Note in particular the crucial (intuitively motivated) restrictions on the use of dummy names in the rules (∀I) and (∃E).

Pre-Fregean logic had serious difficulties coping with arguments that involve both connectives and quantifiers, or which involve propositions with multiple quantifiers. We saw how modern quantificational logic can cope smoothly and naturally with such arguments.

One key bit of practical advice in proof-construction: instantiate existential quantifiers – to start subproofs for use in applications of (∃E) – before instantiating relevant universal quantifiers.

Exercises 33

(a) Render the following inferences into suitable QL languages and provide derivations of the conclusions from the premisses in each case:

(1) Some people are boastful. No one likes anyone boastful. Therefore some people aren't liked by anyone.

(2) There's someone such that if *they* admire some philosopher, then I'm a Dutchman. So if everyone admires some philosopher, then I'm a Dutchman.

(3) Some good philosophers admire Frank; all wise people admire any good philosopher; Frank is wise; hence there is someone who both admires and is admired by Frank.

(4) Only rational people with good judgement are logicians. Those who take some creationist myth literally lack good judgement. So logicians do not take any creationist myth literally.

(5) Everyone who loves or is loved by someone loves themself. There's someone who is loved by someone. Therefore someone loves themself.

(6) If anyone speaks to anyone, then someone introduces them; no one introduces anyone to anyone unless they know them both; everyone speaks to Frank; therefore everyone is introduced to Frank by someone who knows him. [Use R to render ① *introduces* ② *to* ③.]

(7) Any elephant weighs more than any horse. Some horse weighs more than any donkey. If a first thing weighs more than a second thing, and the second thing weighs more than a third, then the first weighs more than the third. Hence any elephant weighs more than any donkey.

(8) Given any two people, if the first admires Gödel and Gödel admires the second, then the first admires the second. Gödel admires anyone who has understood *Principia*. There's someone who has understood *Principia* who admires Gödel. Therefore there's someone who has understood *Principia* who admires everyone who has understood *Principia*!

(b) Give proofs warranting the following QL inferences. (It is between you and your logical conscience how many shortcuts you use!)

(1) $\exists x\forall y\forall z\,Sxyz \therefore \exists x\,Sxxx$
(2) $\exists x\,Sxxx \therefore \exists x\exists y\exists x\,Sxyz$
(3) $\exists x(Fx \to \forall y\,Gy) \therefore (\forall x\,Fx \to \forall x\neg Gx)$
(4) $\forall y(Fm \lor Lny), \neg Fm \therefore \forall z\,Lnz$
(5) $\forall x(P \lor Fx) \therefore (P \lor \forall xFx)$
(6) $\forall x(Gx \lor Hx), \forall y(Gy \to Ryy), \forall z(Hz \to Rzz) \therefore \forall x\exists y(Rxy \land Ryx)$
(7) $\exists x\forall yLxy, \exists x\exists y(Myx \land Gx), \forall y\forall z((Lyz \land Gz) \to \neg Hy) \therefore \exists x\neg Hx$
(8) $(\forall xFx \to \exists yGy) \therefore \exists x\exists y(Fx \to Fy)$

(c) Give proofs showing that the following are QL theorems:

(1) $\forall x\exists y(Fx \to Fy)$
(2) $(\forall x\,Fx \lor \exists x\neg Fx)$
(3) $\exists x(Fx \to \forall yFy)$
(4) $\exists x\forall y(\neg Fy \lor Fx)$
(5) $\forall x\exists y(\forall z\,Sxxz \to \forall z\,Sxyz)$

(d*) Although all our examples of QL proofs so far start from zero or more sentences and end with a sentence, we won't build that into our official characterization of QL proofs – they can go from wffs involving dummy names to a conclusion which may involve a dummy name.

Say that the wffs Γ (not necessarily all sentences) are QL-*consistent* if there is no QL proof using wffs from Γ as premisses and ending with '$\perp$' – compare Exercises 22(d*). Show that

(i) If $\Gamma, \alpha(\tau)$ are QL-inconsistent, then we can derive $\neg\alpha(\tau)$ from Γ.
(ii) If $\Gamma, \alpha(\tau)$ are QL-inconsistent and Γ include $\forall\xi\alpha(\xi)$ then Γ themselves are already QL-inconsistent.

Conclude that

(1) If the wffs Γ are QL-consistent and $\forall\xi\alpha(\xi)$ is one of them, then $\Gamma, \alpha(\tau)$ are also QL-consistent for any term τ.

and further show that

(1*) If the wffs Γ are QL-consistent and $\forall\xi\alpha(\xi)$ is one of them, and $\tau_1, \ldots, \tau_k$ are terms that appear in Γ, then $\Gamma, \alpha(\tau_1), \alpha(\tau_2), \ldots, \alpha(\tau_k)$ are also QL-consistent.

Show similarly that

(2) If the wffs Γ are QL-consistent and $\exists\xi\alpha(\xi)$ is one of them, then $\Gamma, \alpha(\tau)$ are also QL-consistent if τ is a term that doesn't appear in Γ.

Also show these simpler results:

(3) If the wffs Γ are QL-consistent and $\neg\forall\xi\alpha(\xi)$ is one of them, then $\Gamma, \exists\xi\neg\alpha(\xi)$ are QL-consistent.
(4) If the wffs Γ are QL-consistent and $\neg\exists\xi\alpha(\xi)$ is one of them, then $\Gamma, \forall\xi\neg\alpha(\xi)$ are QL-consistent.