6 First-order Peano Arithmetic

The previous chapter introduced two weak theories of arithmetic, BA and Q. In this chapter – jumping over a whole family of possible intermediate strength theories – we introduce a *much* richer first-order theory of arithmetic, PA. It's what you get by adding a generous *induction principle* to Q. But what's that?

6.1 Induction: the very idea

Here is the basic idea we need:

> Whatever numerical property we take, if we can show that (i) zero has that property, and also show that (ii) this property is always passed down from a number n to its successor Sn, then this is enough to show (iii) the property is passed down to *all* numbers.

This is the key informal *principle of (arithmetical) induction*, and is a standard method of proof for establishing arithmetical generalizations.

It is plainly a sound rule. Why? If a property is possessed by zero and then is passed down from each number to its successor, then it must percolate down to any given number – since you can get to any number by starting from zero and repeatedly adding one.

For those not so familiar with this standard procedure for establishing arithmetical generalizations, let's have a mini-example of the principle at work in an everyday informal mathematical context. We'll take things in plodding detail.

Suppose we want to establish that *the sum of the first* n *numbers is* $n(n+1)/2$. Well, let's first introduce some snappy notation.

$$\chi(n) \text{ abbreviates the claim:} \quad 1+2+3+\ldots+n = n(n+1)/2$$

Then (i) trivially $\chi(0)$ is true (the sum of the first zero numbers is zero)!

And now suppose that $\chi(n)$ is true. Then it follows that

$$\begin{aligned} 1+2+3+\ldots+Sn &= (1+2+3+\ldots+n)+(n+1) \\ &= n(n+1)/2+(n+1) \\ &= (n+1)(n+2)/2 \\ &= (Sn)(Sn+1)/2 \end{aligned}$$

Which means that $\chi(Sn)$ will be true too. Hence, generalizing, we have (ii) for all numbers n, if $\chi(n)$, then $\chi(Sn)$

Given (i) and (ii), by induction we can conclude $\forall n\chi(n)$ – i.e., as we claimed, for any n, the sum of the first n numbers is $n(n+1)/2$.

Here's another example of same principle at work, in telegraphic form. Suppose we want to show that all the theorems of a certain Hilbert-style axiomatized propositional calculus are tautologies.

This time, define $\chi(n)$ to be true if the conclusions of proofs (without non-logical premisses) up to n steps long are tautologies. We note that $\chi(0)$ is true (trivial!). And then we can argue that if $\chi(n)$ then $\chi(Sn)$ (for note that the last step of an $n+1$-step proof must either be another instance of an axiom, or follow by modus ponens from two earlier conclusions which – since $\chi(n)$ is true – must themselves be tautologies, and either way we get another tautology). Hence, 'by induction on the length of proofs', we get the desired result.[1]

6.2 The induction axiom, the induction rule, the induction schema

The intuitive idea, then, is that for any property of numbers, if zero has it and it is passed from one number to the next, then all numbers have it. How are we going to implement this idea in a formal theorem of arithmetic?

We've just expressed the intuitive idea as a generalization over properties of numbers. Hence to frame a corresponding formal version, it might seem natural to use a formalized language that enables us to generalize not just over numbers but over properties of numbers. This means it might seem natural to use a language with *second-order* quantifiers. That is to say, we not only have first-order quantifiers running over all the numbers, but also a further sort of quantifier which runs over all properties-of-numbers. In such a language, we could state a second-order

Defn. 22. Induction Axiom.

$$\forall \mathsf{X}(\{\mathsf{X0} \land \forall \mathsf{x}(\mathsf{Xx} \to \mathsf{XSx})\} \to \forall \mathsf{x}\mathsf{Xx})$$

[1] Beginners, in week one of their first logic course, have the contrast between 'deductive' and 'inductive' arguments dinned into them. So sternly are they drilled to distinguish conclusive deductive arguments from merely probabilistic inductions, that some students can't help feeling uncomfortable when they first hear of 'induction' being used in arithmetic!

So let's be clear. We have a case of *empirical, non-conclusive, induction*, when we start from facts about a limited sample (observed swans, say) and infer a claim about the whole population (e.g. concluding that all swans are white). The gap between the sample and the whole population, between the particular bits of evidence and the universal conclusion, allows space for error. The inference isn't deductively watertight.

By contrast, in the case of *arithmetical induction*, we start not from a bunch of claims about particular numbers but from an already universally quantified claim about all numbers, something of the form: *for all n, if $\chi(n)$ then $\chi(Sn)$*. We put that universal claim together with the particular claim $\chi(0)$ to derive another universal claim, for all n, $\chi(n)$. This time, then, we are going from universal to universal, and that's how it is possible for there to be no deductive gap.

(Predicates are conventionally written upper case: so too for variables that are to occupy predicate position.) You can read this Axiom as saying " for any property X, given that 0 has X, and given that if a number has X so does its succcessor, then *every* number has property X."

Seemingly natural though this might be, however, we will be focusing on formal theories whose logical apparatus involves only regular first-order quantification. This isn't due to some perverse desire to work with one hand tied behind our backs. It is because there are some troublesome questions about using second-order logic. For a start, there are technical issues: second-order consequence (at least on the natural understanding) can't be captured in a nice formalizable logical system: so theories using a full second-order logic aren't effectively axiomatizable. And then there are more philosophical issues: just how well do we really understand the idea of quantifying over 'all properties of numbers'? Is that a determinate totality which we can quantify over? We don't want to tangle with these worries here and now.

However, if we don't have second-order quantifiers available to range over properties of numbers, how can we handle induction? Well, one way is to adopt the following rule of inference:

Defn. 23. Induction Rule. *For any suitable open wff* $\varphi(\mathsf{x})$ *of our arithmetical language, given* $\varphi(\mathsf{0})$ *and* $\forall\mathsf{x}(\varphi(\mathsf{x}) \to \varphi(\mathsf{Sx}))$*, we can infer* $\forall\mathsf{x}\varphi(\mathsf{x})$*.*

We will discuss what counts as 'suitable' in the next section.

Alternatively, we can trade in our induction rule for a general axiom schema, and say:

Defn. 24. Induction Schema. *For any suitable open wff* $\varphi(\mathsf{x})$ *of our arithmetical language, the corresponding instance of this schema*

$$\{\varphi(\mathsf{0}) \land \forall\mathsf{x}(\varphi(\mathsf{x}) \to \varphi(\mathsf{Sx}))\} \to \forall\mathsf{x}\varphi(\mathsf{x})$$

can be taken as an axiom.

It evidently doesn't matter whether we use the Rule with a particular predicate in place of φ, or take the corresponding instance of the Schema and then apply modus ponens: the effect will be the same (assuming our theory can handle conjunctions and conditionals!).

6.3 Being generous with induction?

Suppose then that we start again from the first-order arithmetic Q, and aim to build a richer theory in its language L_A by adding induction, e.g. by adopting all the axioms which are 'suitable' instances of the Induction Schema. But what makes for a 'suitable' predicate for use in an instance of the Schema?

Intuitively, such an instance of the Induction Schema should be acceptable as an axiom, so long as we replace $\varphi(\mathsf{x})$ in the schema by a suitable open wff which expresses a genuine arithmetical property. Why? Well, consider *any* open wff $\varphi(\mathsf{x})$ of L_A. This will be built from no more than the constant term '0', the familiar successor, addition and multiplication functions, plus identity and other logical apparatus. Therefore – you might very well suppose – it ought to express a perfectly determinate arithmetical property (even if, in the general case, we can't always decide whether a given number n has the property or not). *So why not be generous and allow any open L_A wff at all to be substituted for $\varphi(\mathsf{x})$ in the Induction Schema?*

Here's a positive argument for generosity. Suppose for a moment $\varphi(\mathsf{x})$ has no free variables other than 'x'. Then a corresponding instance of the Induction Schema will only allow us to derive $\forall\mathsf{x}\varphi(\mathsf{x})$ when we can *already* prove the corresponding (i) $\varphi(\mathsf{0})$ and also can prove (ii) $\forall\mathsf{x}(\varphi(\mathsf{x}) \to \varphi(\mathsf{Sx}))$. But if we can *already* prove (i) and (ii), we then we can already prove each and every one of $\varphi(\mathsf{0})$, $\varphi(\mathsf{S0})$, $\varphi(\mathsf{SS0})$, ... However, there are no 'stray' numbers which aren't denoted by some numeral; so that means that we can prove of each and every number, taken separately, that φ is true of it. What more can it possibly take for φ to express a genuine property that indeed holds for every number, so that (iii) $\forall\mathsf{x}\varphi(\mathsf{x})$ is true?

In sum, it seems that we can't overshoot by allowing instances of the induction schema for *any* open wff φ of L_A with one free variable. The only *usable* instances from our generous range of induction axioms will be those where we can prove the antecedents (i) and (ii) of the relevant conditionals: and in those cases, we'll have every reason to accept the consequents (iii) too.

In fact, we will officially extend suitable candidates for φ a step further. For we will allow uses of the inference rule where $\varphi(\mathsf{x})$ also has slots for additional variables dangling free (variables which are carried along for the ride, so to speak). Equivalently, we will take the induction axioms to be instances of the induction schema where the expression substituted for $\varphi(\mathsf{x})$ can have other free variables dangling free. For more explanation, see *IGT2*, §§9.3 and 12.1. But we needn't fuss here about elaborating this point.

6.4 First-order Peano Arithmetic introduced

Suppose then that we are generous with induction and agree that *any* open wff of L_A can be used in the induction schema. This means moving on from Q, and jumping right over a range of technically possible intermediate theories, to adopt the much richer theory of arithmetic that we can briskly define as follows:

Defn. 25. PA – **First-order Peano Arithmetic**[2] – *is the theory with a standard first-order logic whose language is L_A and whose axioms are those of* Q *plus the* all *instances of the induction schema that can be constructed from open wffs of L_A.*

[2] The name is conventional. Giuseppe Peano did publish a list of axioms for arithmetic in 1889. But they weren't first-order, only explicitly governed the successor relation, and – as he acknowledged – had already been stated by Richard Dedekind.

Like BA then, PA as presented here has an infinite number of axioms. However that's fine: it is plainly still decidable whether any given wff has the right shape to be one of the new axioms, so this is still a legitimate formalized theory.[3]

Let's have three initial examples of what we can formally prove using induction! First, we'll check that we have plugged the particular gap we noted in Q. Recall: Q has $\forall x(x + 0 = x)$ as an axiom, so that's trivially a theorem of the theory; but it feebly can't prove $\forall x(0 + x = x)$. But PA can. We just put $0 + x = x$ for $\varphi(x)$, prove $\varphi(0)$ (trivial!), prove $\forall x(\varphi(x) \to \varphi(Sx))$, and use induction to conclude $\forall x \varphi(x)$. Spelling that out in painful detail:

1.	$0 + 0 = 0$	Instance of Q's Axiom 4
2.	$\quad 0 + a = a$	Supposition
3.	$\quad S(0 + a) = Sa$	From 2 by the identity laws
4.	$\quad 0 + Sa = S(0 + a)$	Instance of Q's Axiom 5
5.	$\quad 0 + Sa = Sa$	From 3, 4
7.	$0 + a = a \to 0 + Sa = Sa$	From 1, 6 by Conditional Proof
8.	$\forall x(0 + x = x \to 0 + Sx = Sx)$	From 7, since a was arbitrary.
9.	$0 + 0 = 0 \land \forall x(0 + x = x \to 0 + Sx = Sx)$	From 1, 8
10.	$\{0 + 0 = 0 \land \forall x(0 + x = x \to 0 + Sx = Sx)\} \to \forall x(0 + x = x)$	Instance of Induction Schema
11.	$\forall x(0 + x = x)$	From 9, 10 by MP

For a second example, let's note that PA proves $\forall x(x \neq Sx)$. Just take $\varphi(x)$ to be $x \neq Sx$. Then PA trivially proves $\varphi(0)$ because that's Q's Axiom 1. And PA also proves $\forall x(\varphi(x) \to \varphi(Sx))$ by contraposing Axiom 2. And then an induction axiom tells us that if we have both $\varphi(0)$ and $\forall x(\varphi(x) \to \varphi(Sx))$ we can deduce $\forall x \varphi(x)$, i.e. no number is a self-successor. It's as simple as that.

Yet this trivial little result is worth noting when we recall our deviant interpretation which makes the axioms of Q true while making $\forall x(0 + x = x)$ false: that interpretation featured Kurt Gödel himself added to the domain as a rogue self-successor. A bit of induction, however, rules out self-successors.

A third observation. PA allows, in particular, induction for the formula

$$\varphi(x): \quad (x \neq 0 \to \exists y(x = Sy)).$$

But now note that the corresponding $\varphi(0)$ is a trivial logical theorem. Likewise, $\forall x \varphi(Sx)$ is an equally trivial theorem, and that entails $\forall x(\varphi(x) \to \varphi(Sx))$. So we can use an instance of the Induction Schema inside PA to derive $\forall x \varphi(x)$. But that's just Axiom 3 of Q. So our initial presentation of PA – as explicitly having all the Axioms of Q plus the instances of the Induction Schema – involves a certain redundancy.

[3] OK. So PA as we've presented it has an infinite number of axioms: but can we find a finite bunch of axioms for the theory, i.e. a finite set of axioms with the same consequences? No. First-order Peano Arithmetic is not finitely axiomatizable. That's *not* an easy result though!

6.5 Summary overview of PA

Given its very natural motivation, PA is the benchmark axiomatized first-order theory of basic arithmetic. Just for neatness, then, let's bring together all the elements of its specification in one place.

First, to repeat, the *language* of PA is L_A, a first-order language whose non-logical vocabulary comprises just the constant '0', the one-place function symbol 'S', and the two-place function symbols '+', '×', and whose intended interpretation is the obvious one.

Second, PA's deductive *proof system* is some standard version of classical first-order logic with identity. The differences between various presentations of first-order logic of course don't make a difference to what sentences can be proved in PA. It's convenient, however, to fix officially on a Hilbert-style axiomatic system for later metalogical work theorizing about the theory.

And third, its non-logical *axioms* – eliminating the redundancy we just noted from our original specification – are the following sentences:

Axiom 1. $\forall x(0 \neq Sx)$

Axiom 2. $\forall x \forall y(Sx = Sy \to x = y)$

Axiom 3. $\forall x(x + 0 = x)$

Axiom 4. $\forall x \forall y(x + Sy = S(x + y))$

Axiom 5. $\forall x(x \times 0 = 0)$

Axiom 6. $\forall x \forall y(x \times Sy = (x \times y) + x)$

plus every instance of the following

Induction Schema $[\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(Sx))] \to \forall x \varphi(x)$

where $\varphi(x)$ is an open wff of L_A that has 'x' free.[4]

6.6 What can PA prove?

Even BA is good at proving quantifier-free equations. Q adds some ability to prove quantified wffs. We have so far noted just three additional quantified theorems that PA can prove. Though it gets tedious, more exploration will reveal that other familiar and not-so-familiar basic truths about the successor, addition, multiplication functions and about the ordering relation (as defined in §5.7) are provable in PA using induction. So how much more can PA prove?

A great deal! In fact, so much that we might reasonably have hoped – at least before we'd heard of Gödel's incompleteness theorems – that PA would turn out to be a *complete* theory that indeed pins down all the truths of L_A.

[4]Fine print: if $\varphi(x)$ is replaced by a wff with other free variables then we'll need to 'bind' this instance with universal quantifiers, if we prefer every axiom to be a closed sentence.

Something else that would have encouraged this false hope, pre-Gödel. Suppose we define the language L_P to be L_A without the multiplication sign. Take P – so-called Presburger Arithmetic – to be the theory couched in the language L_P, whose axioms are Q's now familiar axioms for successor and addition, plus (the universal closures of) all instances of the induction schema that can be formed in the language L_P. In short, P is PA minus multiplication. *Then* P *is a negation-complete theory of successor and addition.* We are not going to be able to prove that – the argument uses a standard model-theoretic method called 'elimination of quantifiers' which isn't hard, and was known in the 1920s, but it would just take too long to explain.

So the situation is as follows, and was known before Gödel got to work. (i) There is a complete formal axiomatized theory BA whose theorems are all the truths about successor, addition and multiplication expressible in the quantifier-free language L_B. (ii) There is another complete formal axiomatized theory – equivalent to PA minus multiplication – whose theorems are exactly the first-order truths expressible using just successor and addition. Against this background, Gödel's result that adding multiplication in order to get full PA gives us a theory which is incomplete and incompletable (if consistent) comes as a rather nasty surprise. It wasn't obviously predictable that multiplication would make all the difference. Yet it does. In fact, as we've said before, as soon we have an arithmetic as strong as Q which has multiplication as well as addition, we get incompleteness.

And by the way, it isn't that a theory of multiplication must in itself be incomplete. In 1929, Thoralf Skolem showed that there is a complete theory for the truths expressible in a suitable first-order language with multiplication but lacking addition or the successor function. So why then does putting multiplication together with addition and successor produce incompleteness? The answer will emerge shortly enough, but pivots on the fact that an arithmetic with all three functions available can express/capture *all* 'primitive recursive' functions. But we'll have to wait to the next-but-one chapter to explain what *that* means.

6.7 Non-standard models of PA?

We saw that Q has 'a non-standard model', i.e. there is a deviant unintended interpretation that still makes the axioms of Q true. Let's finish the chapter by asking whether PA similarly has non-standard models; does it have deviant unintended interpretations that still make *its* axioms all true?

Yes. Assuming PA is true of the natural numbers (and so is consistent), the Löwenheim-Skolem theorem tells us that it must have non-standard models of all infinite sizes. So PA doesn't pin down uniquely the structure of the natural numbers. Indeed, even if we assume that we are looking for a *countable* model – i.e. a model whose elements could in principle be numbered off – there can be non-standard models of PA. A standard *compactness* argument shows this.

I'll finish this chapter by giving the speedy proof of the last claim. But this is an optional extra. If you don't know about compactness arguments, don't worry – just skip, as nothing in later chapter depends on you knowing this proof.

Suppose, then, that we add to the language of PA a new constant c, and add to the axioms of PA the additional axioms $\mathsf{c} \neq 0$, $\mathsf{c} \neq \bar{1}$, $\mathsf{c} \neq \bar{2}$, ..., $\mathsf{c} \neq \bar{\mathsf{n}}$, ... Evidently each *finite* subset of the axioms of the new theory has a model (assuming PA is consistent and has a model): just take the standard model of arithmetic and interpret c to be some number greater than the maximum n for which $\mathsf{c} \neq \bar{\mathsf{n}}$ is in the given finite suite of axioms.

Since each finite subset of the infinite set of axioms of the new theory has a model, the compactness theorem tells us the whole theory must have a model. And then, by the downward Löwenheim-Skolem theorem there will be in particular a *countable* model of this theory, which contains a zero, its successors-according-to-the-theory, and rogue elements including the denotation of c. Since this structure is a countable model of PA-plus-some-extra-axioms it is, a fortiori, a countable model of PA, and must be distinct from the standard one as its domain has more than just the zero and its successors.

"OK: that was smart! But can you now describe one of these countable-but-weird models of PA? Being a countable model, we can take its domain to be the numbers again: but what do the interpretations of the successor, addition and multiplication functions now look like?" Good question. And it is indeed difficult to describe these models, for a principled reason. *Tennenbaum's Theorem* tells us that, for any non-standard model of PA, the interpretations of the addition and the multiplication functions can't be nice computable functions, and so can't be arithmetical functions that we can give a familiar sort of description of. That indeed makes it a bit tricky to get a direct handle on the non-standard models.